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Application of Picard-Padé Technique for Obtaining the Exact Solution of 1-D Hyperbolic Telegraph Equation and Coupled System of Burger's Equations

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Abstract

In this Letter, we introduce a modification of the Picard iteration method (PIM) using Laplace transform and Padé approximation to obtain closed form of solutions of certain parabolic and hyperbolic nonlinear partial differential equations (NPDEs). Special attention to study the convergence analysis of the proposed method is given. Convergence analysis is reliable enough to estimate the maximum absolute error of the solution given by PIM. Some test examples such as Blowup in finite time, the viscous Burger's equation with chemical reaction, the coupled system of Burger's equations and the one dimensional hyperbolic telegraph equation are given. The results obtained ensure that this procedure is a powerful tool for solving large amount of problems in physics and engineering.

Keywords: Picard iteration method; Padé approximation; Convergence analysis.

Introduction

Many different methods have recently introduced to solve nonlinear problems, such as, variational iteration method ([1], [5], [9], [18], [19], [21]), Adomian decomposition method ([2], [24], [25]) and homotopy perturbation method ([20], [22], [23]). The Adomian decomposition method provides solutions as a series by employing the so-called Adomian's polynomials which are related to the derivatives of the nonlinearities; therefore, these nonlinearities must be analytical functions of the dependent variables and this has often been ignored in the literature, for the existence and uniqueness of solutions to, for example, initial-value problems in ordinary differential equations is ensured under much milder conditions ([10], [14]). However, the decomposition method may be formulated in a manner that does not require that the nonlinearities be differentiable with respect to the dependent variables and their derivatives [15]. Other techniques that also require that the nonlinearities be analytical functions of the dependent variable and provide either convergent series or asymptotic expansions to the solution include perturbation methods [13], the homotopy perturbation technique and the homotopy analysis procedure [23].

The second-order telegraph equation with constant coefficients, models mixture between diffusion and wave propagation by introducing a term that accounts for effects of finite velocity to standard heat or mass transport equation [6]. However, this equation is commonly used in signal analysis for transmission and propagation of electrical signals [12] and also has applications in other fields (see [17] and the references therein). In recent years, much attention has been given in the literature to the development, analysis, and implementation of stable methods for the numerical solution of second-order hyperbolic equations (see [8] and the references therein). By way of contrast, iterative techniques for solving a large class of linear or nonlinear differential equations without the tangible restriction of sensitivity to the degree of the nonlinear term and also it reduces the size of calculations besides, its interactions are direct and straightforward. These techniques include the well-known Picard fixed-point iterative procedure.

In this paper we present a modification of PIM, this modification depends on the Padé approximates [4], Laplace transform and Taylor series method [24]. Special attention to study the convergence analysis of the proposed method is given. Convergence analysis is reliable enough to estimate the maximum absolute error of the solution given by PIM. To guarantee this study, effectively employ this modification to a certain class of parabolic and hyperbolic nonlinear partial differential equations (NPDEs) Therefore, this modification of PIM has been widely used in solving nonlinear problems to overcome the shortcoming of other methods.

The rest of this paper is organized as follows: In Section 2, the basis idea of the Picard iteration method is given. In section 3, special attention to study of convergence analysis is introduced. In section 4, the basic idea of the Padé approximation on the series solution is given. in section 5, the modified algorithm of PIM is introduced. In section 6, illustrative examples are given. In section 7, the PIM-Padé-technique to 1-D hyperbolic telegraph equation is presented. In section 8, the PIM-Padé-technique to coupled system of Burger's is presented. The conclusions and discussion are presented in section 9.

Picard iteration method

To illustrate the analysis of PIM, we limit ourselves to consider the following nonlinear differential equation in the type:

$$Lu = Ru + N(u), \tag{1}$$

with specified initial conditions, where $L = \frac{\partial}{\partial t}$ and R is linear bounded operator i.e., it is possible to find a number $m_1 > 0$ such that $\|Ru\| \le m_1 \|u\|$. The nonlinear term $N(u)$ is Lipschitz continuous with

$$|N(u) - N(v)| < m_2 \; |u - v|, \forall\, t \in J = [0,T], \text{ for any constant } m_2 > 0.$$

The PIM gives the possibility to write the solution of Eq.(1) in the following iteration formula:

$$u_p = u_0 + \int_0^t [\, Ru_{p-1} + N(u_{p-1})\,]\, d\tau, \quad p \ge 1. \tag{2}$$

The successive approximations, u_p, $p \ge 1$, of the solution u will be readily obtained upon using any selective function u_0. The initial values of the solution are usually used for selecting the zeroth approximation u_0. Consequently, the exact solution may be obtained by using:

$$u = \lim_{p \to \infty} u_p. \tag{3}$$

Study of convergence analysis

In this section, the sufficient conditions are presented to guarantee the convergence of PIM, when applied to solve NPDEs, where the main point is that we prove the convergence of the recurrence sequence ([3], [7], [11]), which is generated by using PIM.

Lemma 1

Let $A: U \to V$ be a bounded linear operator and let $\{u_p\}$ be a convergent sequence in U with limit u, then $u_p \to u$ in U implies that $A(u_p) \to A(u)$ in V .

Proof. Since

$$\| Au_p - Au \|_V = \| A(u_p - u) \|_V \le \| A \| \| u_p - u \|_U.$$

Hence

$$\lim_{p \to \infty} \| Au_p - Au \|_V \le \| A \| \lim_{p \to \infty} \| u_p - u \|_U = 0,$$

implies that

$$A(u_p) \to A(u).$$

Theorem 1: (Uniqueness theorem)
The nonlinear problem (1) has a unique solution, whenever $0 < \alpha < 1$, where, $\alpha = (m_1 + m_2)\,T$, where the constants m_1 and m_2 are defined above.

Proof: Since the solution of Eq.(1) can be take the following form:

$$u = f(x) + L^{-1}[Ru + N(u)],$$

where the function $f(x)$ is the solution of the homogenous equation $Lu = 0$, and the inverse operator L^{-1} is defined by $L^{-1}(\bullet) = \int_0^t (\bullet) d\tau$.

Now let, u and u^* be two different solutions to (1) then by using the above equation, we get:

$$\begin{aligned} \left| u - u^* \right| &= \left| \int_0^t [\, R(u - u^*) + N(u) - N(u^*) \,]\, dt \right| \\ &\le \int_0^t [\, \left| R(u - u^*) \right| + \left| N(u) - N(u^*) \right| \,]\, dt \\ &\le (\, m_1 \left| u - u^* \right| + m_2 \left| u - u^* \right|)T \\ &\le \alpha \left| u - u^* \right|, \end{aligned}$$

from which we get $(1 - \alpha)\left| u - u^* \right| \le 0$. Since $0 < \alpha < 1$, then $\left| u - u^* \right| = 0$ implies, $u = u^*$ and this complete the proof.

Now, to prove the convergence of the Picard iteration method, we will rewrite Eq.(2) in the operator form as follows:

$$u_p = A[u_{p-1}], \tag{4}$$

where the operator A takes the following form:

$$A[u] = \int_0^t [\, Ru + N(u) \,]\, d\tau. \tag{5}$$

Theorem 2: (Convergence theorem)
Assume that X be a Banach space and $A : X \to X$ is a nonlinear mapping, and suppose that:

$$\|A[u]-A[v]\| \le \alpha \|u-v\|, \qquad \forall\, u, v \in X. \tag{6}$$

Then A has a unique fixed point. Furthermore, the sequence (2) using PIM with an arbitrary choice of $u_0 \in X$, converges to the fixed point of A and:

$$\left\|u_p - u_q\right\| \le \left[\frac{\alpha^q}{1-\alpha}\right] \left\|u_1 - u_0\right\|. \tag{7}$$

Proof: Denoting $(C[J], \|\bullet\|)$ the Banach space of all continuous functions on J with the norm defined by: $\|f(t)\| = \max_{t\in J} |f(t)|$.

We are going to prove that the sequence $\{u_p\}$ is a Cauchy sequence in this Banach space:

$$\begin{aligned}
\left\|u_p - u_q\right\| &= \max_{t\in J}\left|u_p - u_q\right| \\
&= \max_{t\in J}\left|\int_0^t [\,R(u_{p-1} - u_{q-1}) + N(u_{p-1}) - N(u_{q-1})\,]\,d\tau\right| \\
&\le \max_{t\in J}\int_0^t [\,\left|R(u_{p-1} - u_{q-1})\right| + \left|N(u_{p-1}) - N(u_{q-1})\right|\,]\,d\tau \\
&\le \max_{t\in J}\int_0^t [\,(m_1 + m_2)\left|u_{p-1} - u_{q-1}\right|\,]\,d\tau \\
&\le \alpha\left\|u_{p-1} - u_{q-1}\right\|.
\end{aligned}$$

Let, $p = q+1$ then:

$$\left\|u_{q+1} - u_q\right\| \le \alpha\left\|u_q - u_{q-1}\right\| \le \alpha^2\left\|u_{q-1} - u_{q-2}\right\| \le \dots \le \alpha^q\left\|u_1 - u_0\right\|.$$

Hence by the triangle inequality and the formula for the sum of a geometric progression we obtain for $p > q$, we have:

$$\begin{aligned}
\left\|u_p - u_q\right\| &\le \left\|u_{q+1} - u_q\right\| + \left\|u_{q+2} - u_{q+1}\right\| + \dots + \left\|u_p - u_{p-1}\right\| \\
&\le [\alpha^q + \alpha^{q+1} + \dots + \alpha^{p-1}]\left\|u_1 - u_0\right\| \\
&\le \alpha^q\,[1 + \alpha + \alpha^2 + \dots + \alpha^{p-q-1}]\left\|u_1 - u_0\right\| \\
&\le \alpha^q\left[\frac{1-\alpha^{p-q-1}}{1-\alpha}\right]\left\|u_1 - u_0\right\|.
\end{aligned}$$

Since $0<\alpha<1$ so, $1-\alpha^{p-q}<1$ then:

$$\left\|u_p - u_q\right\| \le [\frac{\alpha^q}{1-\alpha}]\left\|u_1 - u_0\right\|.$$

But $\left\|u_1 - u_0\right\| < \infty$ so, as $q \to \infty$ then $\left\|u_p - u_q\right\| \to 0$. We conclude that $\{u_p\}$ is a Cauchy sequence in $C[J]$ so, the sequence converges and the proof is complete.

Theorem 3: (Error estimate theorem)
The maximum absolute error of the approximate solution u_p to problem (1) is estimated to be:

$$\max_{t\in J}\left| u_{exact} - u_p\right| < \beta, \tag{8}$$

where $\beta = \dfrac{\alpha^q T[m_1\left\|u_0\right\| + k]}{1-\alpha}$, $k = \max_{t\in J}\left|N(u_0)\right|$.

Proof: From Theorem 2 inequality (7) we have:

$$\left\|u_p - u_q\right\| \le \frac{\alpha^q}{1-\alpha}\left\|u_1 - u_0\right\|,$$

as $p \to \infty$ then $u_p \to u_{exact}$ and:

$$\left\|u_1 - u_0\right\| = \max_{t\in J}\left|\int_0^t [Ru_0 + N(u_0)]\,d\tau\right|$$
$$\le \max_{t\in J}\int_0^t [\left|Ru_0\right| + \left|N(u_0)\right|]\,d\tau$$
$$\le T[m_1\left\|u_0\right\| + k],$$

so, the maximum absolute error in the interval J is:

$$\left\|u_{exact} - u_p\right\| = \max_{t\in J}\left|u_{exact} - u_p\right| < \beta.$$

This completes the proof.

Now, we present some basic definition of the Padé approximation (PA), which needed in the next sections of the paper.

The Padé approximation on the series solution

The general setup in approximation theory is that a function f is given and that one wants to approximate it with a simpler function g but in such a way that the difference between f and g is small. The advantage is that the simpler function g can be handled without too many difficulties, but the disadvantage is that one loses some information since f and g are different.

Definition 1

When we obtain the truncated series solution of order at least L + M in t by PIM, we will use it to obtain $PA\left[\frac{L}{M}\right](x,t)$ Padé approximation for the solution $u(x,t)$. The Padé approximation ([4], [25]) are a particular type of rational fraction approximation to the value of the function. The idea is to match the Taylor series expansion as far as possible. We denote the $PA\left[\frac{L}{M}\right]$ to $R(x) = \sum_{i=0}^{\infty} a_i x^i$ by:

$$PA\left[\frac{L}{M}\right](x) = \frac{P_L(x)}{Q_M(x)}, \tag{9}$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M.

$$P_L(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_L x^L, \quad Q_M(x) = 1 + q_1 x + q_2 x^2 + \ldots + q_M x^M. \tag{10}$$

To determine the coefficients of $P_L(x)$ and $Q_M(x)$, we may multiply (9) by $Q_M(x)$, which linearizes the coefficient equations. We can write out (9) in more detail as:

$$\begin{aligned}
&a_{L+1} + a_L q_1 + \ldots + a_{L-M+1} q_M = 0,\\
&a_{L+2} + a_{L+1} q_1 + \ldots + a_{L-M+2} q_M = 0,\\
&\qquad\ldots\\
&a_{L+M} + a_{L+M-1} q_1 + \ldots + a_L q_M = 0,
\end{aligned} \tag{11}$$

$$\begin{aligned}
&a_0 = p_0,\\
&a_1 + a_0 q_1 = p_1,\\
&a_2 + a_1 q_1 + a_0 q_2 = p_2,\\
&\qquad\ldots.\\
&a_L + a_{L-1} q_1 + \ldots + a_0 q_L = p_L.
\end{aligned} \tag{12}$$

To solve these equations, we start with Eq.(11), which is a set of linear equations for all the unknown q's. Once the q's are known, then Eq.(12) gives an explicit formula for the unknown p's, which complete the solution.

Each choice of L, degree of the numerator and M, degree of the denominator, leads to an approximant. The major difficulty in applying the technique is how to direct the choice in order to obtain the best approximant. This needs the use of a criterion for the choice depending on the shape of the solution. A criterion which has worked well here is the choice of $\left[\frac{L}{M}\right]$ approximation such that L = M. We construct the approximation by built-in utilities of Mathematica in the following sections. If PIM truncated Taylor series of the exact solution with enough terms and the solution can be written as the ratio of two polynomials with no common factors, then the Padé approximation for the truncated series provide the exact solution. Even when the exact solution cannot be expressed as the ratio of two polynomials, the Padé approximation for the PIM truncated series usually greatly improve the accuracy and enlarge the convergence domain of the solutions.

The modified algorithm of PIM

In spite of the advantages of PIM, it has some drawbacks. By using PIM, we get a series, in practice a truncated sequence solution. The series often coincides with the Taylor expansion of the true solution at point $x = 0$, in the initial value case. Although the series can be rapidly convergent in a very small region, it has very slow convergence rate in the wider region we examine and the truncated series solution is an inaccurate solution in that region, which will greatly restrict the application area of the method. All the truncated series solutions have the same problem. Many examples given can be used to support this assertion [24].

In this section, we present a modification of PIM by using the Padé approximation and then apply this modification to some systems of NPDEs such as, system of Burger's equations and the one dimensional hyperbolic telegraph equation. The suggested modification of PIM can be done by using the following algorithm.

Algorithm

Step 1: Solve the differential equation using standard PIM;

Step 2: Truncate the obtained sequence solution by using PIM;

Step 3: Take the Laplace transform of the truncated series;

Step 4: Find the Padé approximation of the previous step;

Step 5: Take the inverse Laplace transform.

This modification often gets the exact solution of the differential equation with high accuracy and enlarge the convergence domain of the truncated Taylor series and can improve greatly the convergence rate of the truncated Taylor series.

Now, we implement this algorithm to some examples of linear and nonlinear differential equations to illustrate our modification.

Illustrative examples

Example 1: (Blowup in finite time)

In this example, we consider the following equation [16]:

$$u_t(t) = u^2, \tag{13}$$

subject to $u(0)=1$ which has the exact solution $u(t)=(1-t)^{-1}$ and, therefore, blows up at t = 1.

Step 1: Application of Picard iterative method to this equation:

$$u_{n+1}(t) = u(0) + \int_0^t u_n^2(\tau)\, d\tau, \tag{14}$$

yields the following iterates:

$$u_0(t) = 1,$$

$$u_1(t) = 1 + t,$$

$$u_2(t) = 1 + t + t^2 + \frac{1}{3}t^3,$$

$$u_3(t) = 1 + t + t^2 + t^3 + \frac{2}{3}t^4 + \frac{1}{3}t^5 + \frac{1}{9}t^6 + \frac{1}{36}t^7,$$

$$u_4(t) = 1 + t + t^2 + t^3 + t^4 + \frac{13}{15}t^5 + \frac{2}{3}t^6 + \frac{29}{36}t^7 + \frac{71}{252}t^8 + \frac{86}{567}t^9 + \ldots + \frac{1}{59535}t^{15},$$

and so on, other components can be obtained in a like manner.

Step 2: Truncate the obtained sequence solution by using PIM;

Therefore, the approximate solution can be readily obtained by $u(t) \cong u_4(t)$, which coincide until fifth term with the partial sum of the Taylor series of the solution u(t) at t = 0. Figure 1, shows the error between the exact solution $u(t)$ and $u_4(t)$. From this figure we can conclude that the error at $t \in [0, 0.5]$ is nearly to 0, but at

$t \in [0.5,1]$, the error takes large values.

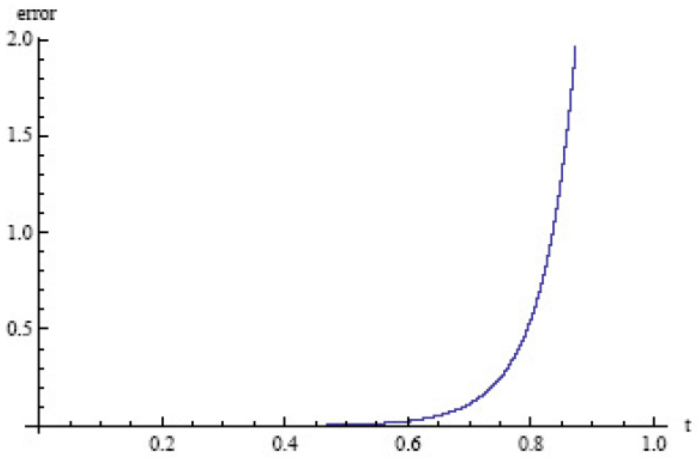

Figure 1: The error between the exact solution u(t) and the solution by PIM, $u_4(t)$.

From these calculations we can see that the coefficient of t will settled in $u_n(t)\,(n>1)$ also the coefficient of t^2 will settled in $u_n(t)\,(n>2)$ and the coeffcient of t^3 will settled in $u_n(t)\,(n>3)$. So the truncated series, we used in calculating the $\frac{L}{M}$ Padé approximation, must have accurate coefficients of t^n to obtain more accurate results. In our calculations we will use the truncated series $u_4(t)$ to order t^4 : $\tilde{u}_4(t) = 1+t+t^2+t^3+t^4$, which coincides with the first five terms in $u_4(t)$, and is a partial sum of the Taylor series of the solution at t = 0.

Step 3: Find the Padé approximation of the previous step; All $\frac{L}{M}$ of the Padé approximation of the above equation with L > 1, M > 1 and $L+M <= 4$ and yields we obtain the exact solution: $\left[\frac{L}{M}\right](\tilde{u}_4(t)) = \frac{1}{1-t}$.

Example 2: (The viscous Burger's equation with chemical reaction)
This example corresponds to the following equation [16]:

$$u_t + uu_x + u(u+2) = u_{xx}, \tag{15}$$

subject to the initial condition: $u(x,0) = e^{-x}$.

Step 1: Application of Picard iterative method to this equation, we obtain:

$$u_{n+1}(x,t) = u(x,0) + \int_0^t [\, u_{nxx}(x,\tau) - u_n u_{nx} - u_n(u_n + 2)\,]\, d\tau, \tag{16}$$

yields the following iterates:

$$u_0(x,t) = e^{-x},$$
$$u_1(x,t) = e^{-x}(1-t),$$
$$u_2(x,t) = e^{-x}(1 - t + \frac{t^2}{2}),$$
$$u_3(x,t) = e^{-x}(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}),$$
$$u_4(x,t) = e^{-x}(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!}),$$
$$u_5(x,t) = e^{-x}(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!}),$$

and so on, other components can be obtained in a like manner.

Step 2: Truncate the obtained sequence solution by using PIM; Therefore, the approximate solution can be readily obtained by, $u(x,t) \cong u_5(x,t)$ which is the partial sum of the Taylor series of the solution $u(x,t)$ at t = 0.

Step 3: Take the Laplace transform of the truncated series;

We apply Laplace transform to $u_5(x,t)$, which yields:

$$£\ [u_5(x,s)] = e^{-x}[\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4} + \frac{1}{s^5} - \frac{1}{s^6}].$$

For the sake simplicity, let $s = \frac{1}{t}$, then: £

$$[u_5(x,t)] = e^{-x}[t - t^2 + t^3 - t^4 + t^5 - t^6].$$

Step 4: Find the Padé approximation of the previous step;

Its $\left[\frac{n+1}{n+1}\right]$ Padé approximation with $n > 0$ yields: $\left[\frac{n+1}{n+1}\right] = e^{-x}\frac{t}{1+t}$.

Relace $t=\frac{1}{s}$, we obtain $\left[\frac{n+1}{n+1}\right]$ in terms of s as follows: $\left[\frac{n+1}{n+1}\right]=e^{-x}\frac{1}{1+s}$.

Step 5: Take the inverse Laplace transform.
By using the inverse Laplace transform to $\left[\frac{n+1}{n+1}\right]$, we obtain the exact solution:

$$u(x,t)=e^{-(x+t)}.$$

PIM-Padé-technique to 1-D hyperbolic telegraph equation

In this section, we implement the proposed algorithm for solving the one-dimensional hyperbolic telegraph equation [6]:

$$u_{tt}+4u_t+2u=u_{xx}, \tag{17}$$

subject to the following initial conditions:

$$u(x,0)=\sin(x), \quad u_t(x,0)=-\sin(x). \tag{18}$$

To apply the presented method for this equation, we use the transformation, $u_t(x,t)=v(x,t)$.

In this case Eq.(17) will reduces to the following coupled system of ordinary differential equations:

$$u_t(x,t)=v(x,t), \tag{19}$$

$$v_t+4v+2u=u_{xx}, \tag{20}$$

with the initial conditions:

$$u(x,0)=\sin(x), \quad v(x,0)=-\sin(x). \tag{21}$$

Now, we solve the system (19)-(20) by using the above algorithm as follows:

Step 1: Solve the system (19)-(20) by using PIM;
According to PIM, we construct the following recurrence formula:

$$u_{n+1}(x,t)=u(x,0)+\int_0^t [v_n(x,\tau]\,d\tau, \tag{22}$$

$$v_{n+1}(x,t)=v(x,0)+\int_0^t [\,u_{nxx}-4v_n-2u_n\,]d\tau. \tag{23}$$

Using the above formula, we can obtain the components of the solution of (19)-

(20)

$$u_0(x,t) = -v_0(x,t) = \sin(x),$$

$$u_1(x,t) = -v_1(x,t) = (1-t)\sin(x),$$

$$u_2(x,t) = -v_2(x,t) = (1-t+\frac{t^2}{2!})\sin(x),$$

$$u_3(x,t) = -v_3(x,t) = (1-t+\frac{t^2}{2!}-\frac{t^3}{3!})\sin(x),$$

$$u_4(x,t) = -v_4(x,t) = (1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!})\sin(x).$$

Therefore, the complete approximate solution can be readily obtained by the same iterative process.

Step 2: Truncate the series solution obtained by PIM;
We have applied the method by using the fourth iteration only, i.e., the approximate solutions are:

$$u(x,t) \cong u_4(x,t) = \sin(x)(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}), \tag{24}$$

$$v(x,t) \cong v_4(x,t) = -\sin(x)(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}). \tag{25}$$

Step 3: Take the Laplace transform of the truncated series;
We apply Laplace transform to $u_4(x,t)$ and $v_4(x,t)$, which yield:

$$£[u_4(x,t)] = -£[v_4(x,t)] = \sin(x)\,(\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s^3}-\frac{1}{s^4}+\frac{1}{s^5}-\frac{1}{s^6}).$$

For the sake simplicity, let $s == \frac{1}{t}$ then:

$$£[u_4(x,t)] = -\,£[v_4(x,t)] = \sin(x)\,(t-t^2+t^3-t^4+t^5-t^6).$$

Step 4: Find the Padé approximation of the previous step;
All of $\left[\frac{L}{M}\right]$ Padé approximation of the above equation with $L>0,\ M>0$ and $L+M<5$, yields: $\left[\frac{L}{M}\right] = \sin(x)\frac{t}{1+t}$. Replace $t=\frac{1}{s}$ we obtain $\left[\frac{L}{M}\right]$ in terms of s as

follows: $\left[\frac{L}{M}\right] = \sin(x)\frac{1}{1+s}$.

Step 5: Take the inverse Laplace transform;

By using the inverse Laplace transform to $\left[\frac{L}{M}\right]$, we obtain the exact solution:

$$u(x,t) = \sin(x)\,e^{-t}, \qquad v(x,t) = -\sin(x)\,e^{-t}.$$

The results of the above example show that our method was capable of solving the problems and generates improves PIM in the convergence rate, and that it often close to their exact solutions.

PIM-Padé-technique to coupled system of Burger's equations

Consider the following system of Burger's equations:

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0, \tag{26}$$

$$v_t - v_{xx} - 2vv_x + (uv)_x = 0, \tag{27}$$

subject to the following initial conditions:

$$u(x,0) = v(x,0) = \sin(x). \tag{28}$$

Now, we solve the system (26)-(27) by using the proposed algorithm as follows:

Step 1: Solve the system (26)-(27) by using PIM;

According to PIM, we construct the following recurrence formula:

$$u_{n+1}(x,t) = u(x,0) + \int_0^t [\, u_{nxx} + 2u_n u_{nx} - (u_n v_{nx} + u_{nx} v_n)\,]\, d\tau, \tag{29}$$

$$v_{n+1}(x,t) = v(x,0) + \int_0^t [\, v_{nxx} + 2v_n v_{nx} - (u_n v_{nx} + u_{nx} v_n)\,]\, d\tau. \tag{30}$$

Using the above formula, we can obtain the components of the solution of (26)-(27)

$u_0(x,t) = v_0(x,t) = \sin(x),$

$u_1(x,t) = v_1(x,t) = (1 - t + 4\cos(x))\sin(x),$

$$u_2(x,t) = v_2(x,t) = (1 - t + \frac{9}{2}t^2 - \frac{8}{3}t^3 + 4t\cos(x) - 4t^2\cos(x) + 12t^3\cos(x) + 12t^2\cos(2x)$$
$$- 8t^3\cos(2x) + \frac{32}{3}t^3\cos(3x) - 8t^2\cos(x))\sin(x).$$

Therefore, the complete approximate solution can be readily obtained by the same iterative process.

Step 2: Truncate the series solution obtained by PIM;
We have applied the method by using the fourth iteration only, i.e., the approximate solutions are:

$$u(x,t) \cong u_4(x,t) = \sin(x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!}), \quad (31)$$

$$v(x,t) \cong v_4(x,t) = \sin(x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!}). \quad (32)$$

The behavior of the error between the exact solution and the obtained solution by PIM in the regions $0 \le x \le 1$ and $0 \le t \le 1$ is shown in figure 2. The numerical results are obtained by using the fourth component only from the formulas (29)-(30). From this figure, we achieved a very good approximation for the solution of the system at the small values of time t, but at the large values of the time t, the error takes large values.

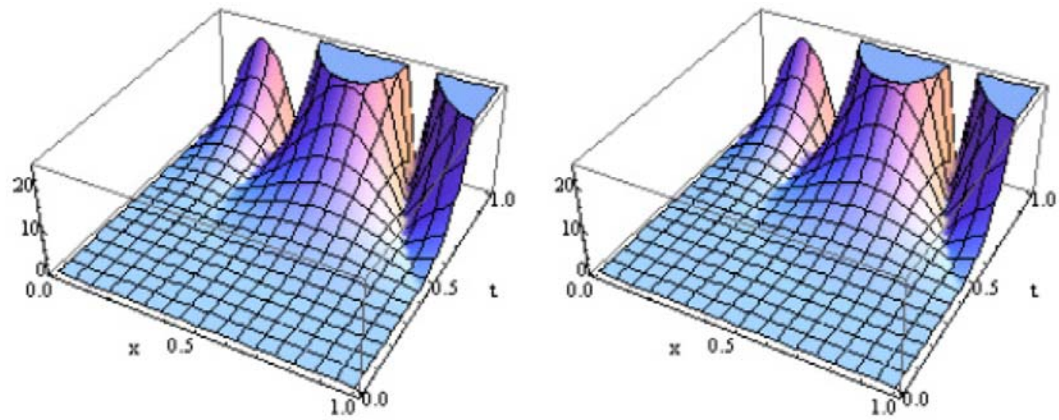

Figure 2: The errors: $erer\_u = |u(x,t) - u_4(x,t)|$ (Left) and erer.

$\_v = |v(x,t) - v_4(x,t)|$ (Right):

Step 3: Take the Laplace transform of the truncated series;

We apply Laplace transform to $u_4(x,t)$ and $v_4(x,t)$, which yield:

$$£[u_4(x,t)] = £[v_4(x,t)] = \sin(x)\,(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4} + \frac{1}{s^5} - \frac{1}{s^6}).$$

For the sake simplicity, let $s == \frac{1}{t}$ then:

$$£[u_4(x,t)] = £[v_4(x,t)] = \sin(x)\,(t - t^2 + t^3 - t^4 + t^5 - t^6).$$

Step 4: Find the Padé approximation of the previous step;

All of the $\left[\frac{L}{M}\right]$ Padé approximation of the above equations with $L > 0, M > 0$ and $L + M < 5$

yields: $\left[\frac{L}{M}\right] = \sin(x)\frac{t}{1+t}.$

Replace $t = \frac{1}{s}$ we obtain $\left[\frac{L}{M}\right]$ in terms of s as follows: $\left[\frac{L}{M}\right] = \sin(x)\frac{1}{1+s}.$

Step 5: Take the inverse Laplace transform;

By using the inverse Laplace transform to $\left[\frac{L}{M}\right]$, we obtain the exact solution:

$$u(x,t) = \sin(x)\,e^{-t}, \qquad v(x,t) = \sin(x)\,e^{-t}.$$

Conclusions and discussion

It has been illustrated that, usually, PIM yields a sequence of iterates which are polynomials in (at least) one of the independent variables and that such a sequence may be numerically useless for values of this independent variable equal to or larger than unity, even though the convergence of these iterates may be ensured by the convergence of the method, see the example 1. It has also been illustrated that, for analytical functions, PIM yields the Taylor series expansion of the solution, and that the convergence of this technique may be substantially increased by analytical continuation, i.e., by dividing the whole interval of integration into disjoint intervals in a piecewise manner. So, in this paper an efficient modification of PIM is presented by using the Padé technique. This modification considerably capable of solving a wide range of linear and nonlinear equations; especially the ones of high nonlinearity order in engineering and physics problems. This purpose was satisfied by solving some nonlinear coupled systems and test examples. The PIM does not need small parameters so that the limitations and non-physical assumptions present in the previous method are eliminated. There- fore, this modification of PIM gives the exact solution of the considered problem and has widely been used in solving nonlinear problems to overcome the shortcoming of other methods such as Adomian decomposition method.

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