Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 7, Number 2 (2011), pp. 163-171

http://www.ripublication.com/gjpam.htm

Covering Graph for Diagram Subgroups of Two Generators of Semigroup Presentation for Words of Length Three

Kalthom Mahmood Alaswad[1] and Abd Ghafur Bin Ahmad[2]

[1]School of Mathematical Science, Faculty of Science and Technology
Al-jabal El-gharbi University, Libya
E-mail: Kaltho76@yahoo.com
[2]School of Mathematical Science, Faculty of Science and Technology
Universiti Kebangsaan Malaysia, Bangi
E-mail: ghafur@.ukm.my

Abstract

For any given diagram group we have a graph. The aim of this paper is to determine all subgroups of diagram group from semigroup presentation $P =< a , b \ : a = b \ >$, to investigate how the diagram group change if we change the subgroup without changing the semigroup presentation.

Keywords: diagram group, covering space, mapping of graphs, semigroup presentation, semigroup.

Introduction

In this section we explain briefly about semigroup presentation and diagram groups which are useful for our purpose. Let $P = < \mathrm{X} \ : r >$ be a semigroup presentation, where X is a set of generator where elements of relations in r is of the form $R_{+1} = R_{-1}$ ($R_{\pm 1}$are reduced positive words on X). We may construct the diagram group $D(P, \mathrm{V})$ where V is a positive word on X as described for example in Abd Ghafur B. and Adel M. [1], Abd Ghafur B. [2], Guba and Sapir [3], Kilibarda [4] and Pride [5].

Let X be set of alphabet. A semigroup presentation P is a pair $< \mathrm{X} \ : r >$where $r \subseteq \mathrm{X}^{+} \times \mathrm{X}^{+}$. An element $x \in \mathrm{X}$ is called generating symbol, while an element $(u, v) \in r$ is called defining relation, and is usually written as $u = v$. The semigroup defined by presentation $< \mathrm{X} \ : r >$ is $\mathrm{X}^{+}/\approx$, where $\approx$ is the smallest congruence on X^{+} containing r in every semigroup presentation $P =< \mathrm{X} \ : r >$

satisfies the following property, if $(u,v) \in r$, then $(v,u) \notin r$. In particular, r does not contain pairs of the form(u,u). If r has relations of the form $(u,1) \in r$ or $(1,v) \in r$ then $P =< X : r >$ is known as a monoid presentation.

Diagram groups are considered from geometrical objects which called semigroup diagrams. These diagrams are drawn and considered as connected graphs. A particular group can be developed from a given graph. This group is known as a diagram group.

Let us give a short summary of the content of this paper. Section two contains the list of the main concepts used in this paper; we introduce the concept of graphs, semigroup presentations, atomic pictures, pictures and diagram groups which are useful for our purpose. In the third section we focus our studies on the new method to study diagram group. We will introduce the main result of this paper, for any given diagram group we have a graph, we construct the graph K_{H_i} where H_i is the smallest normal subgroup of diagram group from semigroup presentation $P =< a , b : a = b >$. And then obtain the covering map $\varphi_H : K_{H_i} \to K_i$.

Basic Definitions

Definition 2.1: A graph K consists of five pair $(V , E, i , \tau, -1)$ where V and E are two disjoint finite sets. Set V is known as the set of vertices while E as the set of edges. Symbols $i , \tau , -1$ are functions $i : E \to V , \tau : E \to V , -1 : V \to V$ such that $i(e) = \tau(e^{-1}) , \tau(e) = i(e^{-1}) , e \neq e^{-1} , e \in E$. The function i and τ are known as the initial and the terminal functions respectively. A graph K is connected if given any two vertices in K, there is a path joining them.

Definition 2.2: A path α in the graph K is a sequence of edges $e_1 e_2 \dots \dots e_n$ where $\tau(e_i) = i(e_{i+1})$ $i = 1,2, \dots \dots , n.$ $(e_i \in E)$. Then $i(\alpha) = i(e_1)$ and $\tau(\alpha) = \tau(e_n)$.

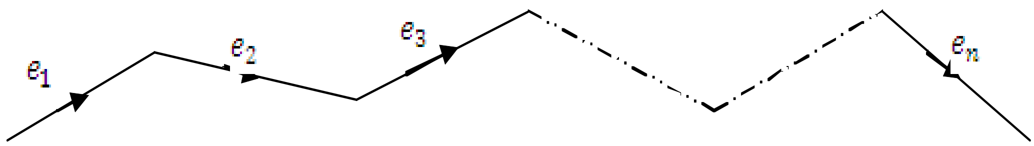

Figure 1: Path of α

We define α to be closed path if $\tau(\alpha) = i(\alpha)$. Let γ and β be two paths in the graph K. If $\tau(\gamma) = i(\beta)$ then the product of γ with β is defined by tracing of γ then followed by β, denoted by $\gamma\beta$.

Theorem 2.3 Let K be a connected graph and fix a vertex v. The algebraic system $\pi_1(K, v) = \{ [\beta] : i(\beta) = \tau(\beta) = v \}$ with binary operation $[\gamma] \cdot [\beta] = [\gamma\beta]$ forms a group called the first fundamental group with base point v where γ , β are closed paths in K. Since the fundamental group of a connected graph is independent of chosen vertex, we simply write $\pi_1(K)$. [7]

Definition 2.4 Let $P = < \text{X} : r >$ be a semigroup presentation. An atomic picture μ over $\mathbb{S}$ is of the form

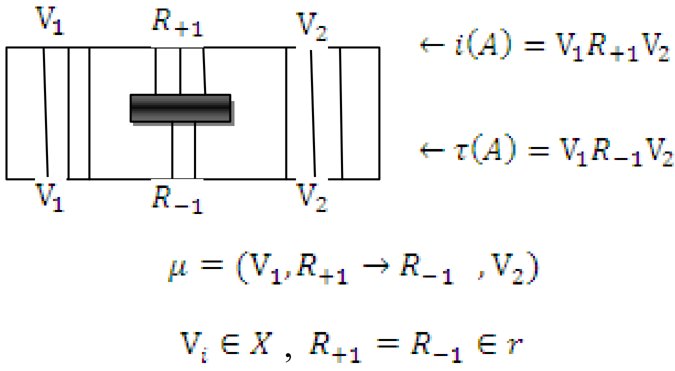

Figure 2: Atomic picture over P.

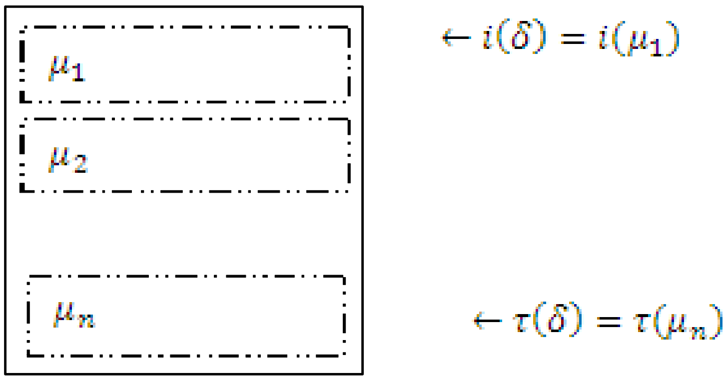

Figure 3: A picture over P.

Definition 2.5 A picture δ over a semigroup presentation P is a collection of atomic pictures $\mu_1, \mu_2, \ldots, \mu_n$such that $\tau(\mu_i) = i(\mu_{i+1})$, $i = 1, \ldots, n-1$.

Definition 2.7 Let $P =< \text{X} : r >$ be a semigroup presentation, consider the following two complex Γ_P:

Vertices: all positive words on X

Edges : all atomic pictures over P

2-cells : all δ pictures over P.

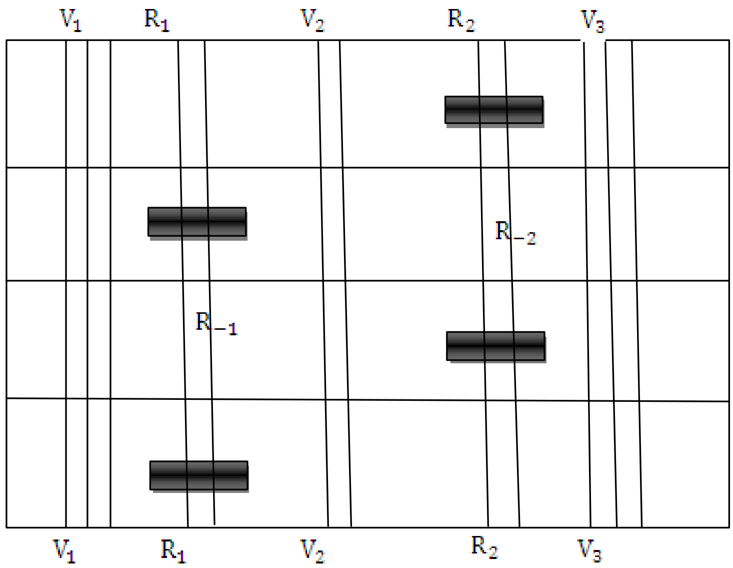

Figure 4: 2-Complex picture.

The fundamental group obtained from this graph is called the diagram group from S with base point V denoted by $D(P, \mathrm{V})$ as described in [4], [5].

Definition 2.8 Let $K_1 = (V_1, E_1, i, \tau, -1)$ and $K_2 = (V_2, E_2, i, \tau, -1)$ be graphs. A mapping $\varphi: K_1 \to K_2$ is a function from $V_1 \cup E_1 \to V_2 \cup E_2$ sending vertices to vertices such that $\varphi(V_1) \subseteq V_2$, edges to edges such that $\varphi(E_1) \subseteq E_2$ and respecting incidences and inversions

$$\varphi\big(i(e)\big) = i\big(\varphi(e)\big), \varphi\big(\tau(e)\big) = \tau\big(\varphi(e)\big), \varphi(e^{-1}) = (\varphi(e))^{-1}.$$

Star of a vertex v is denoted by $star(v) = \{e : e \in E, i(e) = v\}$. The number of edges in $star(v)$ is called the valence of v denoted by $d(v)$.

The mapping $\varphi: K_1 \to K_2$ is locally injective if it is injective on stars, that is $\varphi|_{starv_1}: star(v_1) \to star(\varphi(v_1))$ is injective for each $v_1 \in V_1$. Similarly we may define locally surjective and locally bijective.

Definition 2.9 If $\varphi: \widetilde{K} \to K$ is a locally bijective map and $\widetilde{K}, K$ are connected graphs, then $\widetilde{K}$ is a called a covering graph of K. The mapping φ is called the covering map (covering projection).

Definition 2.10 Let $P = < \mathrm{X} : r >$ be a semigroup presentation, and let $H[\mathrm{S}]$ be a free group on X with basis point V. Let H be a subgroup of $H(\mathrm{S})$. Fix a vertex O in the connected graph $K = (\mathrm{V}, \mathrm{E}, \mathrm{i}, \tau, -1)$. We will construct a connected graph K_H and then the covering map $\varphi_H : K_H \longrightarrow K$ in similar way. Let v be a vertex of K and consider the collection of paths $P_v = \{[\alpha] : i(\alpha) = 0, \tau(\alpha) = v\}$ such that

Vertices $H[\alpha] = \{[\gamma][\alpha] : [\gamma] \in H\}$, $([\alpha] \in P_v , v \in V)$.

Edges all ordered pairs $(H[\alpha], x)$ where x is an edge in Γ such that $\tau(\alpha) = i(x)$.

Functions

i. $i(\mathrm{H}[\alpha], x) = \mathrm{H}[\alpha]$

ii. $\tau(\mathrm{H}[\alpha], x) = \mathrm{H}[\alpha x]$

iii. $(\mathrm{H}[\alpha], x)^{-1} = (\mathrm{H}[\alpha x], x^{-1})$.

Main Results

In this section we determine all subgroups for diagram group from The connected graph K_3. In order to construct the connected graph to obtain the diagram group $D(P, V)$ of P. We only consider the graph, we do not consider the $2-$ cells as described in definition 2.7. We will use this notion $H[\gamma] = H[\beta] \Leftrightarrow [\gamma\beta^{-1}] \in H$ in our work.

Now, for $D(P, V) = K_3$. Vertices of K_3 are $a^3, aba, b^2a, ba^2, a^2b, ab^2, bab, b^3$. 12 edges all together. How to determine covering space of the connected graph K_3 ?

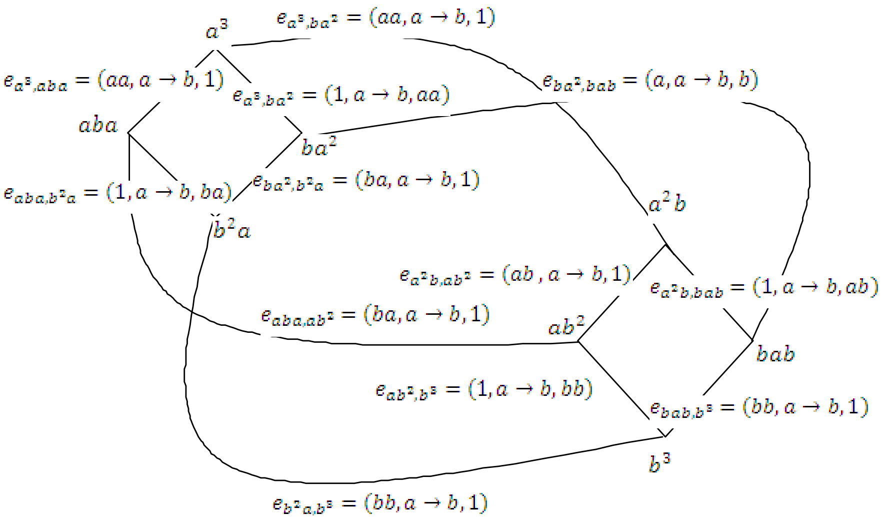

Figure 5: The connected graph K_3.

Lemma 3.1 The component K_H is a connected graph.

Lemma 3.2 The map $\varphi_H : K_H \longrightarrow K$, $\varphi_H(H[\alpha]) = 0$, $\varphi_H(H[\alpha], x) = x$ is a mapping of graphs.

Lemma 3.3 The map $\varphi_H : K_H \longrightarrow K$, $\varphi_H(H[\alpha]) = 0$, $\varphi_H(H[\alpha], x) = x$ is locally bijective.

Theorem 3.4 Consider the connected graph K_3 as is given in figure 5 such that $G=\pi_1(\mathrm{K}_3,a^3)$ containing γ_1 where $\gamma_1=< e_{a^3,aba}, e_{aba,ab^2}, e_{ab^2,a^2b}, e_{a^2b,a^3} >$. If H_{3_1} is the smallest normal subgroup of G containing $<\gamma_1^2>$, then $K_{H_{3_1}}$ is the covering graph for K_3.

Proof. First we will draw easier graph K_3 than the previous one to make sure that the reader will understand.

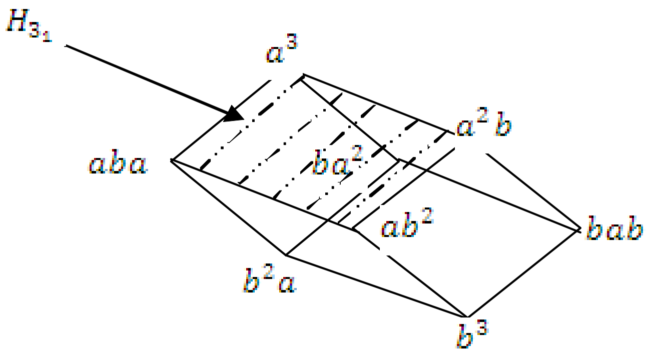

Figure 6: K3.

Fix a vertex a^3 in $K_3=(V,E,\tau,i,-1)$ and let $G=\pi_1(K_3,a^3)$, we will construct the connected graph K_{H_i} and then obtain the covering map $\varphi_H: K_{H_i}\to K_i$ in similar way. Starting by choosing basic $H[\propto]$ where $\propto$ is a path such that $i(\propto)=a^3, \tau(\propto)=v$ for every vertex$v$. Then determine all possible edges $H[1], H[e_{a^3,a^2b}], H[e_{a^3,aba}], H[e_{a^3,ba^2}]$. Since $\varphi_H[H[1]]=a^3$ and $star(a^3)=3$ then $star(H[1])=3$, from $a^3\to a^3$ the vertex in $K_{H_{3_1}}$is $H[1]$, $H[1]$ in $K_{H_{3_1}}$ maps to a^3, from $a^3\to aba$, the vertex in $K_{H_{3_1}}$is $H[e_{a^3,aba}]$ and the edge is $(H[1], e_{a^3,aba})$, $H[e_{a^3,aba}]$ in $K_{H_{3_1}}$maps to aba in K_3. See figure 7.

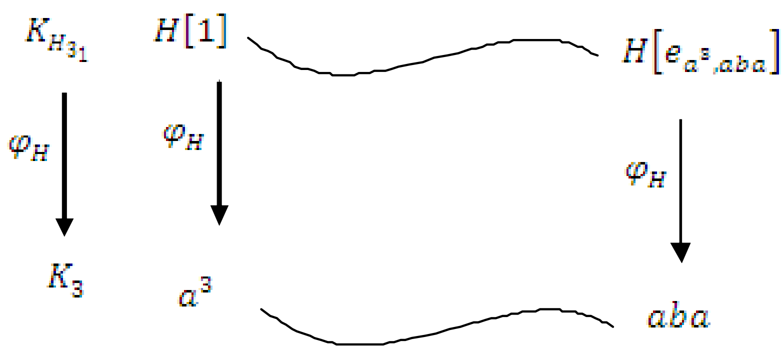

Figure 7

$a^3 \to a^2b$ the vertex in $K_{H_{3_1}}$ is $H[e_{a^3,a^2b}]$ and the edge is $(H[1], e_{a^3,a^2b})$ this vertex in $K_{H_{3_1}}$ maps to a^2b in K_3. See the following table will explain the idea.

From	To	Vertex in $K_{H_{3_1}}$	Vertex in K_3	Edge in $K_{H_{3_1}}$	Edge in K_3
a^3	a^3	$H[1]$	a^3	$-$	$-$
a^3	aba	$H[e_{a^3,aba}]$	aba	$(H[1], e_{a^3,aba})$	$e_{a^3,aba}$
a^3	a^2b	$H[e_{a^3,a^2b}]$	a^2b	$(H[1], e_{a^3,a^2b})$	e_{a^3,a^2b}

Regarding the vertex $H[e_{a^3,aba}]$ should be there are three edges branch out from this vertex, we already recognize one of them$(H[1], e_{a^3,aba})$, the second edge is $(H[1], e_{a^3,aba}e_{aba,ab^2})$, the third one is $(H[1], e_{a^3,aba}e_{aba,b^2a})$where as the vertices are $H[e_{a^3,aba}e_{aba,ab^2}], H[e_{a^3,aba}e_{aba,b^2a}]$ respectively.

From	To	Vertex in $K_{H_{3_1}}$	Vertex in K_3	Edge in $K_{H_{3_1}}$	Edge in K_3
aba	a^3	$H[e_{a^3,aba}]$	a^3	$(H[1], e^{-1}_{a^3,aba})$	$e^{-1}_{a^3,aba}$
aba	ab^2	$H[e_{a^3,aba}e_{aba,ab^2}]$	ab^2	$(H[1], e_{a^3,aba}e_{aba,ab^2})$	e_{aba,ab^2}
aba	b^2a	$H[e_{a^3,aba}e_{aba,b^2a}]$	b^2a	$(H[1], e_{a^3,aba}e_{aba,b^2a})$	e_{aba,b^2a}

And so on. Since the subgroup that we pick is H_{3_1} is the smallest normal subgroup containing $< \gamma_1^2 >$ so to get the diagram groups we will duplicate to get eight vertices namely $H[1]$, $H[e_{a^3,aba}]$, $H[e_{a^3,aba}e_{aba,b^2a}]$, $H[e_{a^3,aba}e_{aba,b^2a}e_{b^2a,ba^2}]$, $H[e_{a^3,aba}e_{aba,b^2a}e_{b^2a,ba^2}e_{ba^2,a^3}]$, $H[e_{a^3,aba}e_{aba,b^2a}e_{b^2a,ba^2}e_{ba^2,a^3}e_{a^3,aba}]$, $H[e_{a^3,aba}e_{aba,b^2a}e_{b^2a,ba^2}e_{ba^2,a^3}e_{a^3,aba}e_{aba,b^2a}]$, $\boldsymbol{H[e_{a^3,aba}e_{aba,b^2a}e_{b^2a,ba^2}e_{ba^2,a^3}e_{a^3,aba}e_{aba,b^2a}e_{b^2a,ba^2}]}$. We will call this diagram$\emptyset_{\Gamma_{H_{3_1}}}$.

Now the first four edges branch out from the first four edges are $(H[1], e_{a^3,aba})$, $(H[e_{a^3,aba}], e_{a^3,a^2b}e_{a^2b,ab^2})$, $(H[e_{a^3,aba}e_{aba,b^2a}], e_{a^3,a^2b}e_{a^2b,ab^2}e_{ab^2,b^3})$, $(H[e_{a^3,aba}e_{aba,b^2a}e_{b^2a,ba^2}], e_{a^3,a^2b}e_{b^2a,ba^2})$. And the other four edges are given in the figure 7. By this notion $H[\gamma] = H[\beta] \Leftrightarrow [\gamma\beta^{-1}] \in H$ we know that $H[e_{a^3,a^2b}] \neq H[e_{a^3,aba}e_{aba,b^2a}e_{b^2a,ba^2}e_{ba^2,a^3}]$ since $e_{a^3,a^2b}e^{-1}_{b^2a,a^3}e^{-1}_{b^2a,ba^2}e^{-1}_{aba,b^2a}e^{-1}_{a^3,aba} \notin H$, also $H[e_{a^3,a^2b}\, e_{a^2b,ab^2}] \neq H[e_{a^3,aba}e_{aba,b^2a}e_{b^2a,ba^2}e_{ba^2,a^3}e_{a^3,aba}]$ since

$e_{a^3,a^2b}\, e_{a^2b,ab^2} e^{-1}_{a^3,\boldsymbol{aba}} e^{-1}_{ba^2,a^3} e^{-1}_{b^2a,ba^2} e^{-1}_{ba^2,a^3} e^{-1}_{\boldsymbol{aba},b^2a} e^{-1}_{e_{a^3,\boldsymbol{aba}}} \notin H$ and so on.

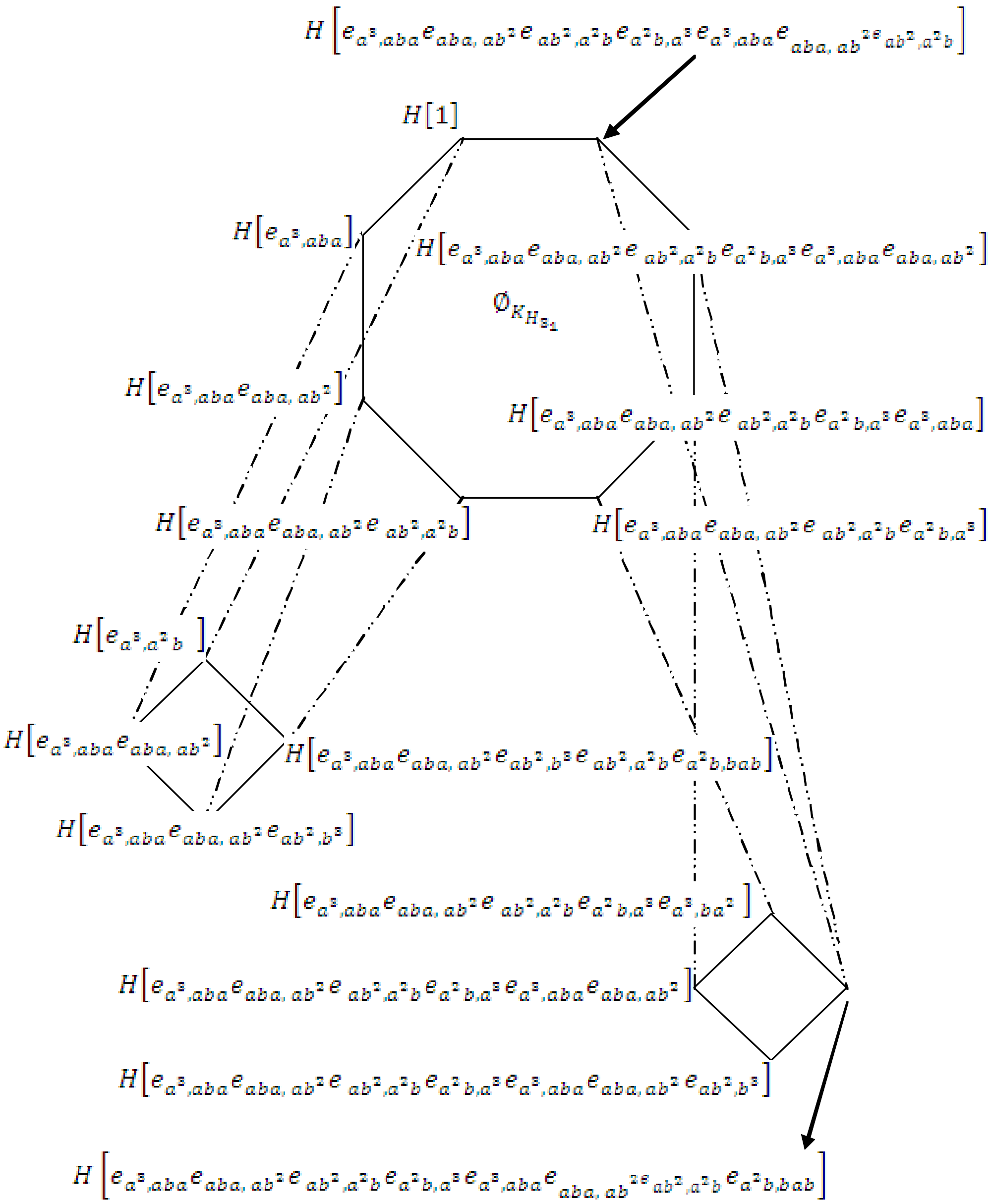

Figure 8: The covering graph K_{H31}

Now let $\varphi_H : K_{H_{3_1}} \longrightarrow K_3$ defined by $\varphi_H(H[\alpha]) = a^3\,, \varphi_H(H[\alpha], x) = x$, we can view this map as locally bijective. Therefore $K_{H_{3_1}}$ is the covering graph for K_3. To sum up If H_{3_n} is the smallest normal subgroup of G containing$< \gamma_1^n >$ then $K_{H_{3_1}}$is the covering graph for K_3. ∎

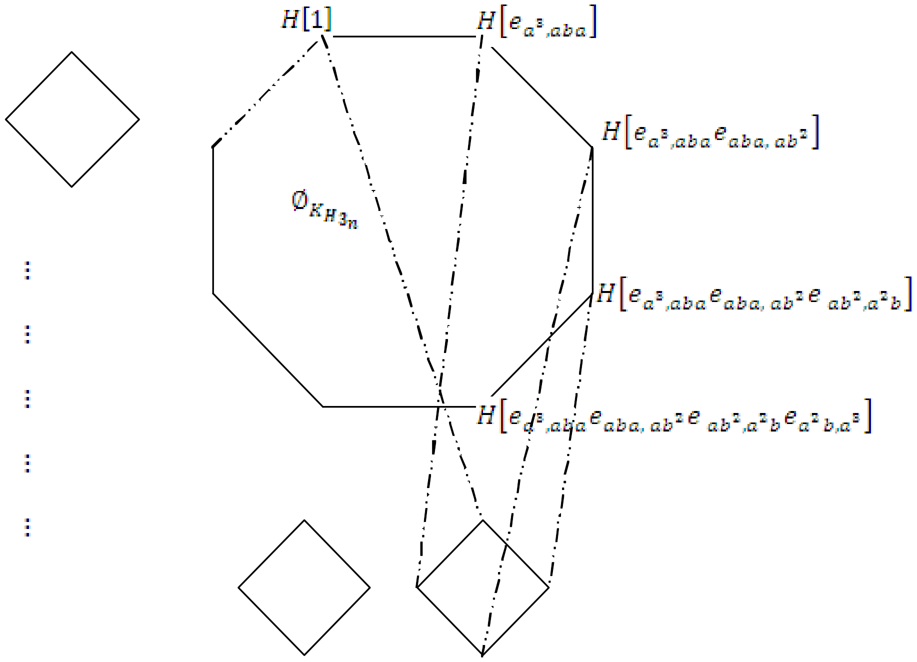

Figure 9: The covering graph K_{H3n}.

Reference

[1] Abd Ghafur B. and Adel M. 2004. The graph of diagram groups constructed from natural numbers semigroup with a repeating generator, J. of Inst. of Math & Com.Sci.(Math.Ser.) 67-69.

[2] Abd Ghafur B. 1995. Triviality Problems For Diagram Groups. In. J. Of Inst. Of Maths. & Comp. Sci. 16(2):105-107.

[3] Guba, V and Sapir, M. 1997. Diagram Groups, Memoirs of the American Mathematical Society 130:1-117.

[4] Kilibarda, V. 1997. On the algebra of semigroup diagrams, int. J of Alg. And comput. 313-338.

[5] Pride, S. 1995. Geometric method in combinatorial group theory, In J. Fountain, editor, Proc. of Int. Conf. on Groups, Semigroups and Formal Languages, Kluwer Publ., 215-232.

[6] Rotman, J. 2002. Advanced Modern Algebra. New Jersey:Pearson Eduction,In.

[7] Rotman, J. 1995. An Introduction to the theory of groups. Forth edition. New York : Springer –Verlag. 377-383.