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On The Lattice of Convex Sets of a Connected Graph

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Abstract

In this paper it is shown that the set of all convex sets of a finite connected graph together with empty set partially ordered by set inclusion forms a lattice. Some of the properties of the lattice so obtained are studied.

Keywords: convex set, connected graph, complete graph, lattice.

MSC subject classification: 06B99.

Introduction

It is known that the lattice of normal subgroups of a group [1], The lattice of sublattices of a lattice [5], the lattice of convex sublattices of a lattice [6], [7], the lattice of subalgebras of a Boolean aigebra [9], and the lattice of convex subgraphs of a directed graph [8] can be applied to study the internal structure of a group, lattice, Boolean algebra and directed graph.

Motivated by the above studies, in this paper we considered the set of all convex sets of a finite connected graph together with the empty set and found that this set also forms a lattice with respect to the partial order $\subseteq$ (set inclusion).

After introducing some basic concepts, notations and stating some fundamental results, in section 2 we have shown that the set of all convex sets of a finite connected graph together with the empty set, forms a lattice with respect to set inclusion with an example.

In section 3, doubly irreducible elements of Con(G) are characterized as pendant vertices of G. Also it is shown that if Con(G) satisfies the lower covering condition, then G must be complete, which gives certain equivalent conditions. In fact Con(G) is distributive, modular, semimodular if and only if G is complete. In section 4, conditions under which Con(G) is complemented are studied. Also it is shown that Con(G) is planar if and only if G is a path. For terminologies and notations used in this paper we refer to [2] and [3].

Preliminaries

Let G be a finite connected graph. V(G) be the vertex set of G. A set $C \subseteq V(G)$ is said to be convex in G if for every two vertices $u , v \in C$, the vertex set of every u-v geodesic is contained in C. For a finite connected graph G, let the set of all convex sets in G together with empty set be denoted by Con(G). Define a binary relation $\leq$ on Con(G) by, for $A, B \in Con(G)$, $A \leq B$ if and only if $A \subseteq B$. Then clearly $\leq$ is a partial order on Con(G). Moreover $< Con(G), \subseteq >$ forms a lattice where for $A , B \in Con(G)$, $A \wedge B = A \cap B$ and $A \vee B = < A \cup B>$, where $< A \cup B >$ is the convex set generated by $A \cup B$ or equivalently the smallest convex set containing $A \cup B$.

For example, the lattice given in Fig.2 represents the lattice $< Con(G), \subseteq >$ of the connected graph G given in Fig.1.

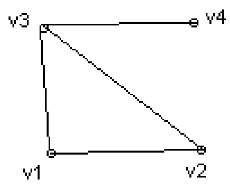

Figure 1

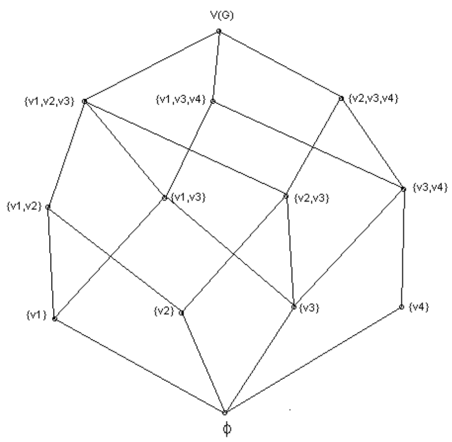

Figure 2

Throughout this paper we use Con(G) to represent the lattice $< Con(G), \subseteq >$

On The Lattice < CON(G), $\subseteq$ >

Remark 3.1: Con(G) is atomic where atoms are convex sets containing only one vertex.

Theorem 3.2: An element A $\in$ Con(G) is doubly irreducible if and only if A = {v} where v is an end point of G (i.e. v is a pendant vertex of G)

Proof: Let A $\in$ Con(G) be doubly irreducible. If A = {v}, where v is a vertex with two edges incident on it, say vv_1 and vv_2. Then A = { v,v_1} $\wedge$ {v, v_2} which implies A is meet reducible, a contradiction. On the other hand if A contains more than one element, say A = { v_1, v_2, v_3, …,v_n } $\in$ Con(G) then A = $\bigvee_{i=1}^{n}\{v_i\}$ and therefore A = {v_i} for some i, since A is join irreducible.

Conversely, for any vertex v in G, A = {v}$\in$ Con(G) is clearly join irreducible by the above remark. If A is meet reducible, say A = B $\wedge$ C = B $\cap$ C for B, C $\in$ Con(G), then v $\in$ B $\cap$ C . Consider u $\in$ B and w $\in$ C where v $\neq$ u and v $\neq$ w. Let v, u_1,u_2, …… ,u be a shortest path connecting v and u in B. Also let v, w_1,w_2, …..,w be a shortest path connecting v and w in C. If u_1 = w_1, then $u_1 \in$ B $\cap$ C contradiction to B $\cap$ C = {v}. Also , if $u_1 \neq w_1$, then {v,u_1} and {v,w_1} are two edges incident on v, contradiction to v is an end point of G. Hence A must be meet irreducible.

Remark 3.3: If a graph G is complete, then Con(G) is a lattice where for A, B $\in$ Con(G), A $\wedge$ B = A $\cap$ B and A $\vee$ B = A $\cup$ B.

Theorem 3.4:
Following statements are equivalent in a connected graph G

1. G is complete
2. Con(G) is distributive
3. Con(G) is modular
4. Con(G) is semimodular
5. Con(G) satisfies lower covering condition.

Proof: (1) $\Rightarrow$ (2) follows from the above remark.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) is true for all lattices.

To prove (5) $\Rightarrow$ (1): Let Con(G) satisfy the lower covering condition. If G is not complete, then there exists two nonadjacent vertices u and v in G. Then ϕ = {u}$\wedge$ {v} $\prec$ {v}. Let u, w_1,w_2,……..,v be a shortest path between u and v. Then {u} < {u,w_1} < {u}$\vee$ {v}which implies {u} $\nprec$ {u} $\vee$ {v} contradiction to Con(G) satisfies lower covering condition. Hence G must be complete.

On the Complementation of CON(G)

Theorem 4.1: If a graph G is complete or a cycle, then Con(G) is complemented. But converse need not be true.

Proof: Clearly if a graph G is complete, then Con(G) is complemented. On the other hand let the graph G be a cycle. Then it is of the form $v_1,v_2,v_3,\ldots\ldots,v_n,v_1$

Case (i): If n is even, then convex sets will be of the form $\{v_1\}$, $\{v_1,v_2\}$, $\{v_1,v_2,v_3\}$, ……………,$\{v_1,v_2,v_3,\ldots\ldots,v_{n/2}\}$ otherwise vertices can be renamed. Complement of any of the above convex sets will be the single vertex set $\{v_{n/2+1}\}$.

Case (ii): If n is odd, then convex sets will be of the form $\{v_1\}$, $\{v_1,v_2\}$, $\{v_1,v_2,v_3\}$, ….,$\{v_1,v_2,v_3,\ldots\ldots, v_{\frac{n+1}{2}}\}$. Complement of $\{v_1,v_2,v_3,\ldots\ldots, v_{\frac{n+1}{2}}\}$ is $\{v_{(\frac{n+1}{2}+1)}$, ………..,$v_n\}$, whereas complement of all other convex sets will be the convex set formed by the edge

$$v_{\frac{n+1}{2}} v_{\frac{n+3}{2}}$$

For all connected graphs with number of vertices less than or equal to 4, converse of the above theorem holds good. The only connected graph with 5 vertices, for which the converse is not true is given by Fig.3. In fact, the lattice Con(G) of this graph is given by Fig.4., which is complemented.

Theorem 4.2: For any connected graph G, Con(G) is planar if and only if the graph is a path.

Proof: Let Con(G) be planar. If the graph is not a path, then either it is a cycle or there exists at least one vertex with minimum three edges incident on it.

Case (i): Let the graph be a cycle. Say v_1, v_2, v_3,……, v_n, v_1. Then Con(G) contains a subposet as shown in Fig 5. Hence it cannot be planar (see [4]).

Case (ii): Let G be a graph having at least one vertex with minimum three edges incident on it as shown in Fig 6.

Then Con(G) contains a subposet as shown in Fig 7. which implies Con(G) cannot be planar (see [4])

Conversely, let the graph be a path v_1, v_2, v_3,……, v_n as shown in Fig. 8. Then Con(G) will be as shown in Fig.9.

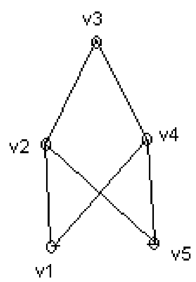

Figure 3

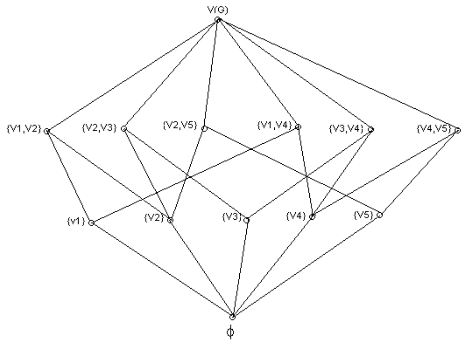

Figure 4

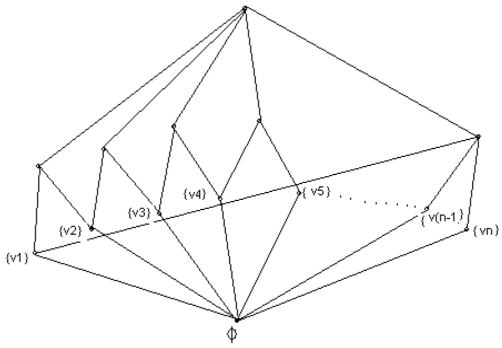

Figure 5

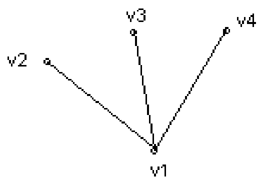

Figure 6

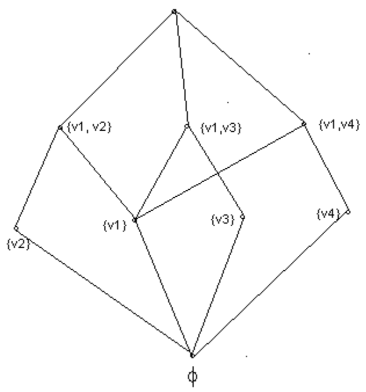

Figure 7

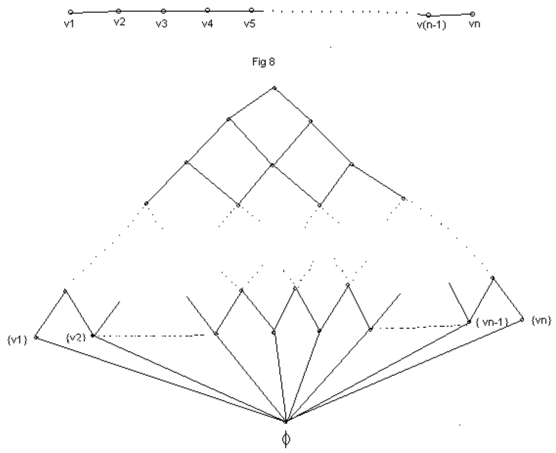

Figure 9

The diagram clearly indicates that it is planar. Hence the proof.

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