Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 7, Number 2 (2011), pp. 129–139

http://www.ripublication.com/gjpam.htm

Generalized r-Permutation and r-Combination Techniques for k-Separable non-inclusion Condition

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Abstract

The purpose of this paper is to present a generalized technique on further restriction on the k-non inclusion condition for r-permutation or r-combination as the case may be, such that a fixed k number of elements ($k \leqslant n$) are always separated in the combinatorial process. We believe the formula to be novel as we have not come across such in literature.

AMS Subject Classification: 05A10.
Keywords: k, r-permutation, k, r-combination, k-separable non-inclusion.

1. Introduction and Preliminaries

Let $X = \{x_i : i = 1, 2, \ldots, n\}$ be a finite collection and let $K = \{x_j : j = 1, 2, \ldots, k\}$ be a sub-collection of X. We consider the problem of selecting r($r \leqslant n$) elements from X in such a way that each selection;

(i) Contains the entire k-elements of the sub-collection K ($r \geqslant k$); we call this the inclusion case. See [4].

(ii) Contains only some part of K and not the entire k-elements; we call this the non-inclusion case. See [3]

(iii) Contains the entire k-elements such that the k-elements are not together; we call this the k-separable inclusion see [2].

(iv) Contains the entire k-elements such that the k-elements are always together; we call this the k-inseparable inclusion.

(v) Contains only some part of K and not the entire k-elements such that the k-elements are not together; we call this the k-separable non-inclusion.

(vi) Contains only some part of K and not the entire k-elements such that the k-elements are always together; we call this the k-inseparable non-inclusion.

Given a finite set X with n elements, in how many ways can we selection its elements with or without respect to the order of selection, such that for any K subset of X the selection will contains only some part of K? The techniques we develop will answer this class of question and a wide variety of other counting problems with further restrictions. In this paper we shall proceed as follows;

Let $X = \{x_i : i = 1, 2, \ldots, n\}$ be a finite collection and let $K = \{x_j : j = 1, 2, \ldots, n\}$ be a sub-collection of X. In this paper we shall proffer solution to problem (v) as stated above. Hence we consider the problem of selecting elements from X in such a way that each selection contains only some part of K and not the entire k-elements such that the k-elements are not together; we call this the k-separable non-inclusion.

The r-permutation and r-combination techniques(see [3],[4])for k non-inclusion condition, becomes a special case for r-permutation and r-combination if $r = k$.

In this paper, we present Mathematical formulae which are rather easy to apply and are applicable in general situations, for the case of combinations and permutations of n distinct elements ($r \leqslant n$) of X with the k-separable non-inclusion of a fixed k number of elements ($k \leqslant r \leqslant n$).

Our formulae are novel, interesting and of general application.

Definition 1.1. see [3] A selection of r elements from $n(r \leqslant n)$ order being (i) significant and (ii) not significant in such that the K sub-collection ($k \leqslant r \leqslant n$) are always included in the r arrangement is called respectively;

i. the k-non-inclusion permutation $P_{ni(n,r,k)}$.

ii. the k-non-inclusion combination $C_{ni(n,r,k)}$.

Definition 1.2. A selection of r elements from a lot of n elements $n(r \leqslant n)$ order being (i) significant and (ii) not significant in such a way that some part a fixed group of k elements and not the entire k elements are always included and are separate in the r arrangement is called respectively;

i. the k-separable non-inclusion permutation $P_{sni(n,r,k)}$.

ii. the k-separable non-inclusion combination $C_{sni(n,r,k)}$.

2. Result

2.1. r-Permutation and r-Combination

Theorem 2.1. Let $X = \{x_i : i = 1, 2, \dots, n\}$, then the r-permutation of n distinct elements ($r \leqslant n$) with the non-inclusion of a fixed k number of elements all at a time ($k \leqslant r \leqslant n$) such that they are always separate is;

$$P_{sni(n,r,k)} = \begin{cases} \sum_{i=0}^{k-1} \frac{P_{(n-k,r-i)} P_{(r-i+1,i)} P_{(k,i)}}{(i)!}, & if\ r \geqslant k\ and\ r+k \leqslant n \\ \sum_{i=r+k-n}^{k-1} \frac{P_{(n-k,r-i)} P_{(r-i+1,i)} P_{(k,i)}}{(i)!}, & if\ r \geqslant k\ and\ r+k > n \\ \sum_{i=0}^{r} \frac{P_{(n-k,r-i)} P_{(r-i+1,i)} P_{(k,i)}}{(i)!}, & if\ r < k\ and\ r+k \leqslant n \\ \sum_{i=r+k-n}^{r} \frac{P_{(n-k,r-i)} P_{(r-i+1,i)} P_{(k,i)}}{(i)!}, & if\ r < k\ and\ r+k > n \end{cases}$$

Proof. If k elements are picked from the set X, then we are left with $(n-k)$ elements. Now, there are three cases to consider; Suppose

i). $k \leqslant r$: An r-permutation can include the entire fixed group ordinarily; we ensure that this dose not happen.

a). $n-k \geqslant r$: The r-arrangement will not include any object from the fixed group. Let each permutation contain i objects from the fixed group with $0 \leqslant i \leqslant k-1$.

If k elements are picked from the set X, then we are left with $(n-k)$ elements.

Step 1
We shall consider (i) and proceed with the following steps for the arrangement of $(n-k)$ element such that the last element can be chosen in $(n-k-r+1)$ ways.

Consequently, by First Counting Principle (FCP) (see [1], [2]), we obtain

$$\frac{(n-k)!}{(n-k-r)!} = P_{(n-k,r)} \tag{2.1}$$

Now, we consider the r-permutation of elements of $(n-k)$ and k simultaneously. For any i elements picked from k elements we will be left to pick $(r-i)$ elements from $(n-k)$ elements. Hence, the possible ways of this arrangement is as follows;

Step 2
For the $(n-k)$ elements, clearly, this elements can be arranged such that the last element

will have $(n-k-r+i+1)$ ways of arrangement.

$$i! = P_{(i,i)} \tag{2.2}$$

Next, we arrange the k element and finally the last k element will be chosen $(k-i+1)$ ways. Consequently, by applying the First Counting Principles (see [1], [2]), we obtain

$$\frac{(n-k)!}{(n-k-r+i)!} = P_{(n-k,r-i)} \tag{2.3}$$

Next, for the separate k elements depending on the i elements picked from it, having picked $(r-i)$ elements from $(n-k)$ elements, then the last element in the arrangement will have $(k-i+1)$ ways of arrangement By First Counting Principles (see [1], [2]), we have

$$\frac{(k)!}{(k-i)!} = P_{(k,i)} \tag{2.4}$$

Step 3
Finally, the i separate positions occupied by k elements will combine with the $(r-i)$ elements from $(n-k)$ elements such that the last element in the arrangement will have $(r-2i+2)$ ways of arrangement.

By First Counting Principles (see [1], [2]), we have

$$\frac{(r-i+1)!}{(r-2i+1)!} = P_{(r-i+1,i)} \tag{2.5}$$

Furthermore, to take care of the initial i arrangement in k elements (identical factors) we must have

$$\frac{P_{(r-i+1,i)}}{(i)!} \tag{2.6}$$

Step 4
By applying First Counting Principles (see [1], [2]) to (2) (3) and (4), we have that r elements of Xcan be arranged as follows:

$$\frac{P_{(n-k,r-i)}P_{(k,i)}P_{(r-i+1,i)}}{(i)!} \tag{2.7}$$

By applying Second Counting Principles (SCP) (see [1], [2]) to (1) and (5), we have

$$P_{sni(n,r,k)} = P_{(n-k,r)} + \sum_{i=1}^{k-1} \frac{P_{(n-k,r-i)}P_{(k,i)}P_{(r-i+1,i)}}{(i)!}$$

$$= \sum_{i=0}^{k-1} \frac{P_{(n-k,r-i)}P_{(k,i)}P_{(r-i+1,i)}}{(i)!}$$

b. $n - k < r$ Every r-permutation must include at least $r + k - n$ objects from the fixed group; observe that $r + k - n < k$. Since otherwise we have $r + k - n \geqslant k \Rightarrow r - n \geqslant o \Rightarrow r \geqslant n$, contradiction.

So each k-permutation contains i objects from the fixed group where $r + k - n \leqslant i \leqslant k - 1$.

So we have $P_{(n-k,r-i)}$ permutation of the $n - k$ objects outside the fixed group and $P_{(k,i)}$ permutation of the i objects in the fixed group of k . Then, by a similar argument we shall have;

$$P_{sni(n,r,k)} = \sum_{i=r+k-n}^{k-1} \frac{P_{(n-k,r-i)}P_{(k,i)}P_{(r-i+1,i)}}{(i)!}$$

ii. $k > r$: Then no r-permutation can include the entire k-fixed group. So that we have $P_{(n,r)}$. However, the $P_{(n,r)}$ set of elements fails to satisfy the separable condition imposed on the k-fixed group. Thus we ensure that this condition is satisfied as follows.

a. $n - k \geqslant k$: We repeat the same steps of argument as obtained i(a) above, but now we have that $k > r$ so that the range of i is such that $i = 0, 1, \ldots, r$. Hence, we have;

$$P_{sni(n,r,k)} = \sum_{i=0}^{r} \frac{P_{(n-k,r-i)}P_{(k,i)}P_{(r-i+1,i)}}{(i)!}$$

b. $n - k < r$: In this case every r-permutation must include at least $r + k - n$ elements from the fixed group, in other-words we need to pick i element from the fixed group such that $n - k \geqslant r - i$. Clearly,$r + k - n \leqslant r$. Since otherwise we have $r + k - n > r \Rightarrow k - n > 0 \Rightarrow k > n, \Rightarrow\Leftarrow$. Thus, each r-permutation for this case contains i elements from the fixed group such that $r + k - n \leqslant i \leqslant r$. Hence, by similar steps of argument, we have;

$$P_{sni(n,r,k)} = \sum_{i=r+k-n}^{r} \frac{P_{(n-k,r-i)}P_{(k,i)}P_{(r-i+1,i)}}{(i)!}$$

the proof is completed. ■

Theorem 2.2. Let $X = \{x_i : i = 1, 2, \ldots, n\}$, then the r-combination of n distinct elements ($r \leqslant n$) with the non-inclusion of a fixed k number of elements all at a time

$(2i-1 \leqslant r \leqslant n)$ such that they are always separate is

$$C_{sni(n,r,k)} = \begin{cases} \sum_{i=0}^{k-1} C_{(n-k,r-i)}C_{(k,i)}, & if\ r \geqslant k\ ,\ r+k \leqslant n\ and\ 2i-1 \leqslant r \\ \sum_{i=r+k-n}^{k-1} C_{(n-k,r-i)}C_{(k,i)}, & if\ r \geqslant k\ ,\ r+k > n\ and\ 2i-1 \leqslant r \\ \sum_{i=r+k-n}^{r} C_{(n-k,r-i)}C_{(k,i)}, & if\ r < k\ ,\ r+k > n\ and\ 2i-1 \leqslant r \\ \sum_{i=0}^{r} C_{(n-k,r-i)}C_{(k,i)}, & if\ r < k\ ,\ r+k \leqslant n\ and\ 2i-1 \leqslant r \end{cases}$$

Proof. Theorem 2.2 follows from the proof of theorem 2.1 by the arrangement in theorem 2.1 it is clear that the $(n-k)$ elements can be arranged in

$$P_{(n-k,r-i)} \quad \text{ways}$$

Thus, to take care of repetition,the possible combination we shall have is

$$\frac{P_{(n-k,r-i)}}{(r-i)!} \quad \text{ways}$$

Now, for the separate i elements picked from k and the $(r-i)$ elements picked from $(n-k)$ combines in such a way that the i elements are separate and generate identical elements. Hence, we must have

$$P_{(r-i+1,i)} = 1 \quad if\ \ 2i-1 \leqslant r$$

(i) So that if we consider the case $k \leqslant r$ for (a) $n-k \geqslant r$, then with a similar argument, we have the following;

$$C_{sni(n,r,k)} = \sum_{i=1}^{k-1} \frac{P_{(n-k,r-i)}C_{(k,i)}}{(r-i)!},$$

$$= \sum_{i=1}^{k-1} C_{(n-k,r-i)}C_{(k,i)},\ if \quad 2i-1 \leqslant r$$

b) $n-k < r$ Every r-permutation must include at least $r+k-n$ objects from the fixed group; observe that $r+k-n < n$ since otherwise $r+k-n \geqslant k \Rightarrow r-n \geqslant 0 \Rightarrow r \geqslant n$, contradiction.

So each k-permutation contains i objects from the fixed group where $r+k-n \leqslant i \leqslant k-1$.

So we have $P_{(n-k,r-i)}$ permutation of the $n-k$ objects outside the fixed group and $P_{(k,i)}$ permutation of the i objects in the fixed group of k . Then, by a similar argument we shall have;

$$C_{sni(n,r,k)} = \sum_{i=r+k-n}^{k-1} C_{(n-k,r-i)}C_{(k,i)}, \ if \quad 2i-1 \leqslant r$$

ii. $k > r$: Then no r-permutation can include the entire k-fixed group. So that we have $P_{(n,r)}$. However the $P_{(n,r)}$ set of elements fails to satisfy the separable condition imposed on the k-fixed group. Thus we ensure that this condition is satisfied as follows.

a. $n-k \geqslant r$: We repeat the same steps of argument as obtained i(a) above, but now we have that $k > r$ so that the range of i is such that $i = 0, 1, \ldots, r$. Hence, we have;

$$C_{sni(n,r,k)} = \sum_{i=0}^{r} C_{(n-k,r-i)}C_{(k,i)}, \ if \ r < k \ , \ r+k \leqslant n \ and \ 2i-1 \leqslant r$$

b. $n-k < r$: In this case every r-permutation must include at least $r+k-n$ elements from the fixed group, in other words we need to pick i element from the fixed group such that $n-k \geqslant r-i$.

Clearly, $r+k-n \leqslant r$. Since otherwise we have $r+k-n > r \Rightarrow k > n, \Rightarrow\Leftarrow$. Thus, each r-permutation for this case contains i elements from the fixed group such that $r+k-n \leqslant i \leqslant r$. Hence, by similar steps of argument, we have;

$$C_{sni(n,r,k)} = \sum_{i=r+k-n}^{r} C_{(n-k,r-i)}C_{(k,i)}, \ if \ r < k \ , \ r+k > n \ \text{ and } \ 2i-1 \leqslant r$$

In this section we consider the case where $r = k$ which is a special case for both the r-permutation and r-combination. We shall provide a formula that shall solve problems of this kind for n distinct elements of X with the non-inclusion of a fixed k number of elements all at a time with given restrictions on k. ■

3. k-permutation and k-combination

Corollary 3.1. Let $X = \{x_i : i = 1, 2, \ldots, n\}$, then the k-permutation of n distinct elements of X with the non-inclusion of a fixed k number of elements all at a time $(k \leqslant n)$ such that they are always separate is;

$$P_{sni(n,k,k)} = \begin{cases} \displaystyle\sum_{i=r+k-n}^{r} \frac{P_{(n-k,k-i)}P_{(k,i)}P_{(k-i+1,i)}}{(i)!} & if \ \ 2k \leqslant n \\ \displaystyle\sum_{i=r+k-n}^{r} \frac{P_{(n-k,k-i)}P_{(k,i)}P_{(k-i+1,i)}}{(i)!} & if \ \ 2k > n \end{cases}$$

Proof. The proof of corollary 3.1 will follow the same argument and steps as that of theorem 2.1. ■

Corollary 3.2. Let $X = \{x_i : i = 1, 2, \ldots, n\}$, then the k-combination of n distinct elements of X with the non-inclusion of a fixed k number of elements all at (once) a time $(2i - 1 \leqslant n)$, such that they are always separate is

$$C_{sni(n,k,k)} = \begin{cases} \sum_{i=0}^{k-1} C_{(n-k,k-i)} C_{(k,i)}, \; if \;\; 2i-1 \leqslant k \; 2k \leqslant n \\ \sum_{i=2k-n}^{k-1} C_{(n-k,k-i)} C_{(k,i)} \;\; if \;\; 2i-1 \leqslant k \; 2k > n \end{cases}$$

Proof. The proof of Corollary 3.2 will follow the same argument and steps as that of theorem 2.2. ■

Corollary 3.3. Let $X = \{x_i : i = 1, 2, \ldots, n\}$, then the n-permutation and n combination of n distinct elements with the non-inclusion of a fixed k number of elements all at (once) a time $(k \leqslant n)$ such that they are always separate is

$$P_{sni(n,k,k)} = 0 \quad for \; all \; k \leqslant n$$

$$C_{sni(n,k,k)} = 0 \quad for \; all \; k \leqslant n$$

Proof. The proof of corollary 3.3 follows from the proof of theorem 2.1 and theorem 2.2. It suffices to put $k = n$, thus, we obtain an impossible situation for both results. ■

Example 3.4. In how many ways can we arrange three letter word from the set $\{A, B, C, D, E\}$ such that C, D and E are not included all at a time and are always separate for cases where order is significant, insignificant.

Solution. Suppose order is significant If C, D and E are not included all at a time picking out $\{C, D, E\}$ we are left with $\{A, B\}$ to form the three letter words, which is nowhere possible, so we have zero value for such way.

If we pick an element from $\{C, D, E\}$ then we must combine it with any two elements of $\{A, B\} i.e. \{C, D, E\} P_{(sni)} \{AB, BA\}$. We observe that;

$$\{C\} P_{(sni)} \{AB, BA\} = \{CAB, ACB, ABC, CBA, BCA, BAC\} = 6 \text{ ways}$$

$$\{D\} P_{(sni)} \{AB, BA\} = \{DAB, ADB, ABD, DBA, BDA, BAD\} = 6 \text{ ways}$$

$$\{E\} P_{(sni)} \{AB, BA\} = \{EAB, AEB, ABE, EBA, BEA, BAE\} = 6 \text{ ways}$$

total number of ways is 18. If we pick any combination of two elements from the set $\{C, D, E\}$ then we must pick an element from $\{A, B\}$ to permute with. i.e

$$\{CD, DC, CE, EC, DE, ED\} P_{(sni)} \{A, B\}$$

Observe that;

$$\{CD\}P_{sni}\{A, B\} = \{CDA, ACD, CDB, BCD\} = 4 \text{ ways}$$

$$\{DC\}P_{sni}\{A, B\} = \{DCA, ADC, DCB, BDC\} = 4 \text{ ways}$$

$$\{CE\}P_{sni}\{A, B\} = \{CEA, ACE, CEB, BCE\} = 4 \text{ ways}$$

$$\{EC\}P_{sni}\{A, B\} = \{ECA, AEC, ECB, BEC\} = 4 \text{ ways}$$

$$\{DE\}P_{sni}\{A, B\} = \{DEA, ADE, DEB, BDE\} = 4 \text{ ways}$$

$$\{ED\}P_{sni}\{A, B\} = \{EDA, AED, EDB, BED\} = 4 \text{ ways}$$

Observe that CD and DE, CE and EC, DE and ED are identical generator with respect to $\{A, B\}$. Thus, all together we have a total of 12 ways.

By SCP, we have 30 ways

Suppose order is **insignificant** we shall have;

$$\{C\}P_{sni}\{AB, BA\} = 1 \text{ way}$$

$$\{D\}P_{sni}\{AB, BA\} = 1 \text{ way}$$

$$\{E\}P_{sni}\{AB, BA\} = 1 \text{ way}$$

Total number of ways is 3

$$\{CD\}P_{sni}\{A, B\} = 2 \text{ ways}$$

$$\{DC\}P_{sni}\{A, B\} = 2 \text{ ways}$$

$$\{EC\}P_{sni}\{A, B\} = 2 \text{ ways}$$

$$\{DE\}P_{sni}\{A, B\} = 2 \text{ ways}$$

$$\{ED\}P_{sni}\{A, B\} = 2 \text{ ways}$$

Total number of ways is 12.

But $\{DE\}$ and $\{DC\}$, $\{CE\}$ and $\{EC\}$, $\{DE\}$ and $\{ED\}$ are identical So we have 6 ways of arrangement.

By SCP, we have 9 ways

Using the **Generalized Techniques**,

Where $n = 5, r = 3, k = 3$

If order is significant, we now apply theorem 2.1 and consider the case when $r + k > n$

$$P_{sni(5,3,3)} = \sum_{i=1}^{2} \frac{P_{(2,3-i)} P_{(3,i)} P_{(4-i,i)}}{(i)!} = 30 \text{ ways}$$

b. Where $n = 5, r = 3, k = 3$
If order is not significant, we now apply theorem 2.2 and consider the case when $r+k > n$

$$C_{sni(5,3,3)} = \sum_{i=1}^{2} C_{(2,3-i)} C_{(3,i)} = 9 \text{ ways}$$

Example 3.5. How many ten letter words can be formed using $\{A, B, C, \ldots, N\}$ such that D, E, G, and K is not included all at a time and must always be separate for the cases (a) order is significant (b) order is insignificant.

Solution. Using the **Generalized Techniques**,
Where $n = 14, r = 10, k = 4$
If order is significant, we now apply theorem 2.1 and consider the case when $r \geqslant k \geqslant$ and $r + k \leqslant n$

$$P_{sni(14,10,4)} = \sum_{i=0}^{3} \frac{P_{(10,10-i)} P_{(4,i)} P_{(11-i,i)}}{(i)!} = 1,745,452,800 \text{ ways}$$

b. Where $n = 14, r = 10, k = 4$
If order is not significant, we now apply theorem 2.2 and consider the case when $r+k \leqslant n$

$$C_{sni(14,10,4)} = \sum_{i=0}^{3} C_{(10,10-i)} C_{(4,i)} = 551 \text{ ways}$$

4. Conclusion

Clearly, as in Example 3.5 we can observe that it is humanly impossible to list all the 1,745,452,800 possible arrangements that will contain D, E,G, and K separately without the use of computer. To know how many ways the elements of the set X can be arranged, we need an elegant mathematical formulae that can be used in place of listing. Hence, the results we obtain will handle problems of this nature.

References

[1] Kenneth H. Rosen, Discr. Math. and its Applications, Singapore. McGraw-Hill, pg. 232–296, 1991.

[2] C. Moore, M. Laisin and O.C. Okoli, Generalized r-permutation and r-combination techniques for k-separable inclusion to appear in International Journal of Applied Mathematics and Statistics (IJAMAS), Vol. 23; Issue No. D11, 2011.

[3] O.C. Okoli, M. Laisin, and W. Uwandu, A generalized K-Permutation and K-Combination techniques for k-non-inclusion condition, Natural and Applied Sciences journal (Vol. 10 No. 1), pg. 36–44, 2009.

[4] O.C. Okoli, M. Laisin, and W. Uwandu, k-partition of k-Permutation and k-Combination for either k-inclusion and k-non-inclusion, Natural and Applied Sciences journal (Vol. 10 No. 2), pg. 138–145, 2009.

[5] Richard, J Discr. Math, New York, Macmillian, pg. 43–65, 1984.

[6] R. Simion and F. Schmidt, Restricted permutations, Europ. Journal Comb. Vol. 6, pg. 383–406, 1985.