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N-Approximately Weak Amenability of the Second Dual of a Banach Algebra

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Abstract

In this paper we investigate n-approximately weak amenability of a Banach algebra $\mathcal{A}$, and show that for $n \geq 2$, n-approximately weak amenability passes from $\mathcal{A}''$ to $\mathcal{A}$, where $\mathcal{A}''$ is the second dual of $\mathcal{A}$ equipped with the first Arens product. Also we prove that under certain condition n-approximately weak amenability inherits by closed subalgebras of $\mathcal{A}$.

Introduction

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. A bounded linear mapping $D: \mathcal{A} \to X$ is said to be a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}).$$

For example, let $x \in X$, and define

$$\delta_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

Then δ_x is a derivation; such derivation is called inner derivation. We denote by $\mathcal{Z}^1(\mathcal{A}, X)$, the space of all continuous derivation from $\mathcal{A}$ into X and by $\mathcal{N}^1(\mathcal{A}, X)$, the space of all inner derivation from $\mathcal{A}$ into X. Also we denote by $\mathcal{H}^1(\mathcal{A}, X)$ the quotient space $\mathcal{Z}^1(\mathcal{A}, X)/\mathcal{N}^1(\mathcal{A}, X)$, which is called the first Banach cohomology group of $\mathcal{A}$ with coefficients in X [5].

The concept of n-weak amenability of Banach algebras are introduced in [6]. A Banach algebra $\mathcal{A}$ is said to be n-weakly amenable if every derivation from $\mathcal{A}$ into n^{th} dual space $\mathcal{A}^{(n)}$ is inner or equivalently, $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$. A derivation $D: \mathcal{A} \to X$ is called (weakly) approximately inner if there exists a net (x_i) in X such that, for all $a \in \mathcal{A}$, $D(a) = \lim_i \delta_{x_i}(a)$ in the norm topology (in the weak topology),

[9].

The Banach algebra $\mathcal{A}$ is approximately amenable if every derivation from $\mathcal{A}$ into every dual $\mathcal{A}$-module X' is approximately inner. In [9], Ghahramani and Loy have introduced the notation of approximate amenability and shows that approximate amenability of $\mathcal{A}''$ implies that of $\mathcal{A}$, see also [8].

The next result which is contained in [10], plays a key role in the sequel.

Proposition 1.1 *A derivation $D: \mathcal{A} \to X$ is approximately inner if and only if it is weakly approximately inner.*

Throughout this paper, the first and second Arens products are denoted by □ and ♦, respectively. These products can be defined as follows

$$\Phi \square \Psi = w^* - \lim_i \lim_j a_i b_j, \quad \Phi \blacklozenge \Psi = w^* - \lim_j \lim_i a_i b_j$$

where (a_i) and (b_j) are nets in $\mathcal{A}$ such that $a_i \to \Phi$ and $b_j \to \Psi$ in w^*-topology. The Banach algebra $\mathcal{A}$ is said to be Arens regular if the products □ and ♦ coincide on $\mathcal{A}''$, see[1] and [7].

We identify a Banach space with its canonical image in its second dual and we use $k_n: \mathcal{A}^{(n)} \to \mathcal{A}^{(n+2)}$ for the canonical embedding. In particular, we denote by k_0 the canonical embedding from $\mathcal{A}$ into $\mathcal{A}''$.

n-Approximately weak amenability

The concept of n-approximately weak amenability of Banach algebras are investigated in [11]. A Banach algebra $\mathcal{A}$ is said to be n-approximately weakly amenable if every derivation from $\mathcal{A}$ into n^{th} dual space $\mathcal{A}^{(n)}$ is approximately inner. If $n = 1$, then we say that $\mathcal{A}$ is approximately weakly amenable. We remark that approximately weak amenability passes from $\mathcal{A}''$ to $\mathcal{A}$ under each of the following conditions [8].

1) $\mathcal{A}$ is a left ideal in $\mathcal{A}''$.
2) $\mathcal{A}$ is dual Banach algebra.
3) $\mathcal{A}$ is regular and every derivation from $\mathcal{A}$ into $\mathcal{A}'$ is weakly compact.

In this paper we study n-approximately weak amenability of $\mathcal{A}$ and show that for $n \geq 2$, n-approximately weak amenability inherits from $\mathcal{A}''$ to $\mathcal{A}$. Also we prove that for each closed subalgebra $\mathcal{B}$ of $\mathcal{A}$, n-approximately weak amenability of $\mathcal{A}$ implies that of $\mathcal{B}$, if there exist closed ideal I of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{B} \oplus I$ as a direct sum.

Lemma 2.1 Let $\mathcal{A}$ be a Banach algebra and $n \in \mathbb{N}$.

(i) If $D: \mathcal{A} \to \mathcal{A}^{(2n)}$ is a derivation, then so is $D'': \mathcal{A}'' \to \mathcal{A}^{(2n+2)}$.

(ii)If $n \geq 2$ and $D: \mathcal{A} \to \mathcal{A}^{(2n-1)}$ is a derivation, then so is

$$(k_{(2n-2)})' \circ D'': \mathcal{A}'' \to \mathcal{A}^{(2n+1)}.$$

Proof. See [2].

Theorem 2.2 *Let $\mathcal{A}$ be a Banach algebra. Then for $n \geq 2$, n-approximately weak amenability of $\mathcal{A}''$ implies that of $\mathcal{A}$.*

Proof. It is enough to prove the theorem for $n = 2m$, $m \in \mathbb{N}$. If $n = 2m+1$, the proof completely similar to the even case. Let $m \in \mathbb{N}$ and $D: \mathcal{A} \to \mathcal{A}^{(2m)}$ be a derivation. Then by preceding lemma D'' is also a derivation. Since $\mathcal{A}''$ is $(2m)$-approximately weakly amenable, $D'' = \lim_i \delta_{a_i^{(2m+2)}}$ for some $a_i^{(2m+2)} \in \mathcal{A}^{(2m+2)}$. In particular,

$$D(a) = \lim_i \delta_{a_i^{(2m+2)}}(a),$$

and regarding $D(a)$ in $\mathcal{A}^{(2m)}$, we have $D = \lim_i \delta_{a_i^{(2m)}}$. Thus D is approximately inner and so $\mathcal{A}$ is $(2m)$-approximately weakly amenable.

For a Banach algebra $\mathcal{A}$, let $\mathcal{A}^{op}$ be the Banach algebra with underlying Banach space same as $\mathcal{A}$ and with product $\circ$ given by $a \circ b = ba$. Then it is easy to check that $\mathcal{A}$ is$n$-approximately weakly amenable amenable if and only if$\mathcal{A}^{op}$is n-approximately weakly amenable amenable.

Now let $\mathcal{A}$ be a commutative Banach algebra. Then for all $\Phi, \Psi \in \mathcal{A}''$, we have $\Phi \square \Psi = \Psi \blacklozenge \Phi$ and so $(\mathcal{A}'', \square) = (\mathcal{A}'', \blacklozenge)^{op}$. It follows that $(\mathcal{A}'', \square)$ is n-approximately weakly amenable if and only if $(\mathcal{A}'', \blacklozenge)$ is n-approximately weakly amenable. A similar fact is valid if there exist a continuous involution on $\mathcal{A}$. Therefore we have the next result.

Proposition 2.3 *Let $\mathcal{A}$ be a Banach algebra which is commutative or there exist a continuous involution on $\mathcal{A}$. Then $(\mathcal{A}'', \square)$ is n-approximately weakly amenable if and only if $(\mathcal{A}'', \blacklozenge)$ is n-approximately weakly amenable.*

Suppose that $\theta: \mathcal{A} \to \mathcal{B}$ is a continuous homomorphism from a Banach algebra $\mathcal{A}$ into a Banach algebra $\mathcal{B}$. Then $\mathcal{B}^{(n)}$ is a Banach $\mathcal{A}$-bimodule by the following module structures

$$(a, b^{(n)}) \mapsto \theta(a) \cdot b^{(n)}, \quad (b^{(n)}, a) \mapsto b^{(n)} \cdot \theta(a) \quad (a \in \mathcal{A}, b^{(n)} \in \mathcal{B}^{(n)}).$$

It is easy to check that the adjoints $\theta^{(n)}$ of θ are $\mathcal{B}$-module morphisms.

Theorem 2.4 *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A} = \mathcal{B} \oplus I$ for some closed ideal I and closed subalgebra $\mathcal{B}$. Then n-approximately weak amenability of $\mathcal{A}$ implies that of $\mathcal{B}$.*

Proof. Let θ be the natural projection from $\mathcal{A}$ onto $\mathcal{B}$ and let J be the natural injection from $\mathcal{B}$ into $\mathcal{A}$. Then clearly θ and J are continuous homomorphisms with $\theta \circ J = I_{\mathcal{B}}$. We prove the theorem for $n = 2m$, $m \in \mathbb{N}$. For the odd case $n = 2m - 1$, the proof is similar. Let $m \in \mathbb{N}$ and let $D: \mathcal{B} \to \mathcal{B}^{(2m)}$ be a continuous derivation. Take $\overline{D} = J^{(2m)} \circ D \circ \theta: \mathcal{A} \to \mathcal{A}^{(2m)}$, then $\overline{D}$ is a derivation and so $\overline{D} = \lim_i \delta_{a_i^{(2m)}}$ fore some $a_i^{(2m)} \in \mathcal{A}^{(2m)}$. Let $x \in \mathcal{B}$, then $\overline{D}(J(x)) = \lim_i (\delta_{a_i^{(2m)}}(J(x)))$ in the weak topology and so for each $\mu \in \mathcal{B}^{(2m+1)}$ we have

$$\begin{aligned}
\langle \mu, D(x) \rangle &= \langle (J^{(2m+1)} \circ \theta^{(2m+1)})(\mu), D(\theta \circ J(x) \rangle \\
&= \langle \theta^{(2m+1)}(\mu),\ J^{(2m)}(D \circ \theta(J(x))) \rangle \\
&= \langle \theta^{(2m+1)}(\mu),\ \overline{D}(J(x)) \rangle \\
&= \lim_i\ \langle \theta^{(2m+1)}(\mu),\ \delta_{a_i^{(2m)}}(J(x)) \rangle \\
&= \lim_i\ \langle \mu,\ \theta^{(2m)}(\delta_{a_i^{(2m)}}(x)) \rangle \\
&= \lim_i\ \langle \mu,\ x \cdot \theta^{(2m)}(a_i^{(2m)}) - \theta^{(2m)}(a_i^{(2m)}) \cdot x \rangle.
\end{aligned}$$

Therefore $D(x) = \lim_i\ (x \cdot \theta^{(2m)}(a_i^{(2m)}) - \theta^{(2m)}(a_i^{(2m)}) \cdot x)$ in the weak topology and so D is weakly approximately inner. Thus D is approximately inner by proposition 1.1. Hence $\mathcal{B}$ is $(2m)$-approximately weakly amenable.

We recall that an element $\Phi_0 \in \mathcal{A}''$ is called mixed unit if it is a right unit for $(\mathcal{A}'', \square)$ and a left unit for $(\mathcal{A}'', \blacklozenge)$. It is well-known that an element $\Phi_0 \in \mathcal{A}''$ is a mixed unit if and only if it is a weak * cluster point of some bounded approximate identity $(e_\alpha)_{\alpha \in I}$ in $\mathcal{A}$, see [3].

Let Φ_0 be a mixed unit for $\mathcal{A}''$. Then by proposition 5.9 of [7], we have

$$\mathcal{A}'' = (\mathcal{A}' \cdot \mathcal{A})' \oplus (\mathcal{A}' \cdot \mathcal{A})^{\perp},$$

where $(\mathcal{A}' \cdot \mathcal{A})^{\perp}$, is the annihilator of $\mathcal{A}' \cdot \mathcal{A}$ in $\mathcal{A}''$. The dual $(\mathcal{A}' \cdot \mathcal{A})'$ of $\mathcal{A}' \cdot \mathcal{A}$ can be identified with the closed subspace $\Phi_0 \square \mathcal{A}''$ of $\mathcal{A}''$ and the mapping $k: \Phi_0 \square \mathcal{A}'' \to (\mathcal{A}' \cdot \mathcal{A})'$ define by $\langle k(\Phi_0 \square \Phi), f \cdot a \rangle = \langle \Phi, f \cdot a \rangle$ is a Banach algebra isomorphism, which is an isometry if $\| \Phi_0 \| = 1$.

Now as an consequence of theorem 2.4, we deduce the next results.

Corollary 2.5 *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity of norm* 1. *Then n-approximately weak amenability of $\mathcal{A}''$ implies that of $(\mathcal{A}' \cdot \mathcal{A})'$.*

Example 2.6 (***i***) *Let G be a locally compact group and $\mathcal{A} = L^1(G)$ be its group algebra. Since $\mathcal{A}$ has an approximate identity bounded by* 1 *and $\mathcal{A}' \cdot \mathcal{A} = LUC(G)$, the space of bounded left uniformly continuous functions on G, therefore by above result n-approximately weak amenability of $\mathcal{A}''$ implies that of $LUC(G)'$.*

$(\boldsymbol{ii})$ Let G be a locally compact discrete group and $\mathcal{A} = l^1(G)$. Then by theorem 3.2 of [4], we have $\mathcal{A}'' = \mathcal{A} \oplus C_0^{\perp}$, as a direct sum, where $C_0^{\perp}$ is a closed two-sided ideal in$\mathcal{A}''$. Thus,n-approximately weak amenability of $\mathcal{A}''$ implies that of $\mathcal{A}$ by theorem 2.4.

Let $\mathcal{A}$ be a 2-weakly amenable Banach algebra. Then $\mathcal{A}$ dose not necessarily have the property that $\mathcal{H}^1(\mathcal{A},\mathcal{A}) = \{0\}$. For example let G be an infinite, compact, non-abelian group and take $\mathcal{A} = L^1(G)$. Then $\mathcal{A}$ is 2-weakly amenable but $\mathcal{H}^1(\mathcal{A},\mathcal{A}) \neq \{0\}$, for details see [6]. The next result shows that for 2-weakly amenable dual Banach algebra $\mathcal{A}$, every derivation on $\mathcal{A}$ must be inner.

Theorem 2.7 *Let $\mathcal{A}$ be a dual Banach algebra such that $\mathcal{A}$ is 2-weakly amenable. Then $\mathcal{H}^1(\mathcal{A},\mathcal{A}) = \{0\}$.*

Proof. Let $d: \mathcal{A} \to \mathcal{A}$ be a continuous derivation and let $D = k_0 \circ d$. Then D is a derivation from $\mathcal{A}$ into $\mathcal{A}''$. Since $\mathcal{A}$ is 2-weakly amenable there exist $\Phi \in \mathcal{A}''$ such that $D(a) = \delta_\Phi(a)$ for each $a \in \mathcal{A}$. Let E be the predual space of $\mathcal{A}$ and suppose that $J: E \to \mathcal{A}'$ be the canonical mapping. Then it is easy to check that J' is a homomorphism from $(\mathcal{A}'', \square)$ onto $\mathcal{A}$. Set $x = J'(\Phi)$, then we have

$$\begin{aligned} d(a) = J'(\widehat{d(a)}) &= J'(D(a)) \\ &= J'(a \cdot \Phi - \Phi \cdot a) \\ &= a \cdot J'(\Phi) - J'(\Phi) \cdot a \\ &= a \cdot x - x \cdot a \end{aligned}$$

Therefore $d(a) = \delta_x(a)$ for all $a \in \mathcal{A}$ and so $d: \mathcal{A} \to \mathcal{A}$ is inner. Thus $\mathcal{H}^1(\mathcal{A},\mathcal{A}) = \{0\}$.

Theorem 2.8 Let $\mathcal{A}$ be a Banach algebra. Then every derivation on $\mathcal{A}$ is approximately inner, in each of the following conditions.

$(i)\mathcal{A}$ is dual and 2-approximately weakly amenable.

$(ii)\mathcal{A}$ is 2-weakly amenable.

Proof. By the similar argument to the proof of above theorem, the assertion (i) is clear. Assume that (ii) hold and let $d: \mathcal{A} \to \mathcal{A}$ be a continuous derivation. Set $D = k_0 \circ d$. Then D is a derivation from $\mathcal{A}$ into $\mathcal{A}''$ and so by assumption there exist $\Phi \in \mathcal{A}''$ such that $D(a) = \delta_\Phi(a)$ for each $a \in \mathcal{A}$. Let (x_i) be a net in $\mathcal{A}$ such that $x_i \to \Phi$ in weak* topology. Then for all $f \in \mathcal{A}'$, we have

$$\begin{aligned}
\langle f, d(a)\rangle &= \langle f, D(a)\rangle \\
&= \langle f, a\cdot\Phi - \Phi\cdot a\rangle \\
&= \langle \Phi, f\cdot a - a\cdot f\rangle \\
&= \lim_i \langle \widehat{x_i}, f\cdot a - a\cdot f\rangle \\
&= \lim_i \langle f, a\cdot x_i - x_i\cdot a\rangle.
\end{aligned}$$

Therefore $d(a) = \lim_i \delta_{x_i}(a)$ in the weak topology of $\mathcal{A}'$. Thus d is weakly approximately inner and so is approximately inner by proposition 1.1.

Let $\mathcal{A} = L^1(G)$, for an infinite, compact, non-abelian group G. Then $\mathcal{A}$ is 2-weakly amenable and so by above theorem every derivation on $\mathcal{A}$ is approximately inner. However, $\mathcal{H}^1(\mathcal{A},\mathcal{A}) \neq \{0\}$.

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