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Constants of Motion Cubic in the Momenta

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Abstract

In this paper, we consider the Hamiltonian systems with two degrees of freedom whose complete integrability is afforded by first integrals cubic in the momenta. A classification from an invarint theory point of view of a family of clssical potentials is obained.

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1. Introduction

Recall that in 1935 Drach [1] listed ten completely integrable Hamiltonian systems defined in Minkowski plane $\mathbb{E}_1^2$. Their complete integrability is afforded by the existence of an additional constant of motion which are cubic in the momenta. In recent years the ten integrable systems of Drach have received much attention from the viewpoint of the theory of superintegrable systems [3–7] (see also the references therein). Thus, it has been shown [10, 13] that seven of the ten Drach potentials are, in fact, superintegrable, admitting in addition a constant of motion that is quadratic in the momenta. The recent developmnet in the invariant theory of Killing tensors (see for example, [8, 9, 11, 12] and the references therein) give new insight to the study of Hamiltonian systems. Smirnov and Yue [12] employs the inductive version of the method of moving frames and finds a complete set of isometry group invariant of Killing tensors of valence three defined in Minknowski plane, which aims at, among others, the study of Drach's potentials. We continue in this direction and classify constants of motion cubic in the momenta using an invariant theory.

2. Hamiltonian systems and constants of motion

Let (M, g) be an m-dimensional pseudo-Riemannian manifold of constant curvature. Consider a Hamiltonian system defined by a natural Hamiltonian of the form (we adopt the Einstein summation convention)

$$H = \frac{1}{2} g^{ij}(\mathbf{q}) p_i p_j + V(\mathbf{q}), \quad i, j = 1, \ldots, m, \tag{2.1}$$

via the canonical Poisson bi-vector $\mathbf{P}_0 = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$, where g^{ij} are the components of the metric tensor g and $(\mathbf{q}, \mathbf{p}) \in T^*M$ (cotangent bundle) are the standard position-momenta coordinates.

The Hamiltonian system with m degrees of freedom is a system of autonomous PDEs

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}, \qquad i = 1, \ldots, m. \tag{2.2}$$

Definition 2.1. A constant of motion of the system (2.2) is a function F in the position-momenta coordinates $(\mathbf{q}, \mathbf{p})$ that remains constant along any trajectory on the phase space.

It follows immediately from Definition 2.1 and the formula (2.2) that

Theorem 2.2. A function F is a constant of motion of the Hamiltonian system (2.2) iff the Poisson bracket of F and H vanishes, that is

$$\{F, H\} = \sum_{i=1}^{m} \left(\frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q^i} \right) = 0.$$

In general the Hamiltonian system with m-degrees of freedom defined by (2.1) may admit constants of motion F which are polynomials in the momenta.

$$F = K_r^{i_1 i_2 \ldots i_r} p_{i_1} p_{i_2} \ldots p_{i_r} + \cdots + K_1^{i_1} p_{i_1} + U, \quad i_1, \ldots, i_r = 1, \ldots, m. \tag{2.3}$$

Definition 2.3. [7] A Hamiltonian system with m degrees of freedom defined by (2.1) is said to be *completely integrable* if there are *m functionally independent* constants of motion which are mutually in involution; The system is called *superintegrable* if there exist more than m functionally independent constants of motion, but not necessarily in involution. If there are exactly $2m - 1$ such constants of motion, the system is called *maximally superintegrable*.

3. Constants of motion cubic in the momenta

Suppose a Hamiltonian system defined by a natural Hamiltonian (2.1) admits a constant of motion that is cubic in the momenta $F = L^{ijk} p_i p_j p_k + K^{ij} p_i p_j + B^i p_i + U$, then the vanishing of the Poisson bracket $\{H, F\} = 0$ takes the following equations expressed in both component and coordinate-free forms respectively [9].

$$L^{(ijk}_{,f} g^{\ell)f} - \frac{3}{2} L^{f(ij} g^{k\ell)}_{,f} = 0 \quad \Leftrightarrow \quad [\mathbf{L}, g] = 0, \tag{3.1}$$

$$K^{(ij}, {}_f g^{k)f} - K^{f(i} g^{jk)}_{,f} = 0 \quad \Leftrightarrow \quad [\mathbf{K}, g] = 0, \tag{3.2}$$

$$B^{(i}_{,f} g^{j)f} - \frac{1}{2} B^f g^{ij}_{,f} - 3L^{fij} V_{,f} = 0 \quad \Leftrightarrow \quad [\mathbf{B}, g] = 3\mathbf{L}\mathrm{d}V, \tag{3.3}$$

$$U_{,f} g^{fi} - 2K^{fi} V_{,f} = 0 \, (U_{,i} = 2K^j{}_i V_{,j}) \quad \Leftrightarrow \quad \mathrm{d}Ug = 2\mathbf{K}\mathrm{d}V, \tag{3.4}$$

$$B^f V_{,f} = 0 \quad \Leftrightarrow \quad \mathbf{B}(V) = 0, \tag{3.5}$$

where $_{,f}$, (,) and [,] denote partial differentiation, symmetrization and the Schouten bracket, respectively. Note $[\mathbf{B}, g] = -\mathcal{L}_{\mathbf{B}}\mathbf{g}$, where $\mathcal{L}$ denotes the Lie derivative operator. It follows immediately from (3.1) and (3.2) that **L** and **K** are Killing tensors of valence three and two respectively, while **B** - in general - is not a Killing vector field.

Equation (3.5) reveals that the potential function V is preserved by the vector field **B**. It is observed [9] next those Equations (3.1)-(3.5) separate into two groups, namely (3.1), (3.3), (3.5) and (3.2), (3.4), involving only components of F which are polynomials of odd order and even order in the momenta respectively. That is, constant of motion can be written as

$$F = F_{odd} + F_{even}, \tag{3.6}$$

where

$$\begin{aligned} F_{odd} &= L^{ijk}(\mathbf{q}) p_i p_j p_k + B^k(\mathbf{q}) p_k, \\ F_{even} &= K^{ij}(\mathbf{q}) p_i p_j + U(\mathbf{q}), \\ \{H, F_{odd}\} &= \{H, F_{even}\} = 0. \end{aligned} \tag{3.7}$$

One immediately arrives at the following result.

Theorem 3.1. A Hamiltonian system with two degrees of freedom determined by a natural Hamiltonian (2.1) which admits a constant of motion (2.3) of order $r \geq 3$ having both even and odd terms in the momenta is necessarily superintegrable, provided that the odd order term and the even order term are functionally independent.

In the following we restrict our attention to a natural Hamiltonian defined on the Minkowski plane $\mathbb{E}^2_1$.

4. Invariant classification of Drach potentials

Drach [1] considered the above problem and posed an ansatz in which the Hamiltonian and an additional constant of motion assume the following forms.

$$H = p_1 p_2 + U, \quad F = -6w\frac{\partial H}{\partial v}p_1 + 6w\frac{\partial H}{\partial u}p_2 - K^{ijk}p_i p_j p_k, \quad q^1 = u, q^2 = v. \tag{4.1}$$

The vanishing of the Poisson bracket indicates that

$$[\mathbf{L}, g] = 0, \tag{4.2}$$

$$[\mathbf{B}, g] = 6\mathbf{L}\mathrm{d}U, \tag{4.3}$$

$$\mathbf{B}(U) = 0. \tag{4.4}$$

The solution to (4.2) (soloving the system of over-determined PDEs) gives the following formulas for the Killing tensors of valence three in Minkowski plane:

$$\begin{aligned} K^{111} &= a_1 - 3a_5 x - 3a_7 x^2 - a_{10}x^3, \\ K^{112} &= (a_3 + a_9 x + a_8 x^2) + y(a_5 + 2a_7 x + a_{10}x^2), \\ K^{122} &= (a_4 - a_9 y - a_7 y^2) + x(a_6 - 2a_8 y - a_{10}y^2), \\ K^{222} &= a_2 - 3a_6 y + 3a_8 y^2 + a_{10}y^3, \end{aligned} \tag{4.5}$$

where K^{111}, K^{112}, K^{122} and K^{222} are the general formulas for the components of the elements of the vector space $\mathcal{K}_0^3(\mathbb{R}_1^2)$. The ten parameters (constants of integration) $a_i, i = 1, \ldots, 10$ represent the dimension $d = \dim \mathcal{K}_0^3(\mathbb{R}_1^2) = 10$. The infinitesimal action of the isometry group $I(\mathbb{R}_1^2)$ on the parameter space Σ_0^3 determined by the parameters $a_i, i = 1, \ldots, 10$ is found to be determined by the vector fields.

Using the newly developed method of moving frames, Simirnov and Yue [12] find a complete set of fundamental isometry invariants, which will be utilized here in the study of Drach potentials.

With the above ansatz, in 1935 Drach [1] derived ten potentials [13].

- Case 1

$$\begin{aligned} U &= \frac{\alpha}{uv} + \beta u^{r_1}v^{r_2} + \gamma u^{r_2}v^{r_1}, \quad \text{where} \quad r_j^2 + 3r_j + 3 = 0, \\ P &= (up_1 - vp_2)^3, \quad w = u^2v^2/2 \end{aligned} \tag{4.6}$$

- Case 2

$$\begin{aligned} U &= \frac{\alpha}{\sqrt{uv}} + \frac{\beta}{(v-\mu v)^2} + \frac{\gamma(v+\mu u)}{\sqrt{uv}(v-\mu u)^2}, \\ P &= 3(up_1 - vp_2)^2(p_1 + \mu p_2), \quad w = uv(v - \mu u), \end{aligned} \tag{4.7}$$

- Case 3

$$U = \alpha uv + \frac{\beta}{(v-au)^2} + \frac{\gamma}{(v+au)^2},$$
$$P = 3(up_1 - vp_2)^2(p_1^2 - a^2p_2^2), \quad w = (v^2 - a^2u^2)/2, \tag{4.8}$$

- Case 4

$$U = \frac{\alpha}{\sqrt{v(u-a)}} + \frac{\beta}{\sqrt{v(u+a)}} + \frac{\gamma u}{\sqrt{u^2 - v^2}},$$
$$P = 3p_2[(up_1 - vp_2)^2 - a^2p_1^2], \quad w = -v(u^2 - a^2), \tag{4.9}$$

- Case 5

$$U = \frac{\alpha}{\sqrt{uv}} + \frac{\beta}{\sqrt{u}} + \frac{\gamma}{\sqrt{v}},$$
$$P = 3p_1p_2(up_1 - vp_2), \quad w = -2uv, \tag{4.10}$$

- Case 6

$$U = \alpha uv + \beta v\frac{2u^2 + c}{\sqrt{u^2 + c}} + \frac{\gamma u}{\sqrt{u^2 + c}},$$
$$P = 3p_2^2(up_1 - vp_2), \quad w = (u^2 - a^2)/2, \tag{4.11}$$

- Case 7

$$U = \frac{\alpha}{(v + 3mu)^2} + \beta(v - 3mu) + \gamma(v - mu)(v - 9mu),$$
$$P = (up_1 + 3mvp_2)^2(p_1 - 3mp_2), \quad w = -m(v + 3mu), \tag{4.12}$$

- Case 8

$$U = (v + mu/3)^{-2/3}[\alpha + \beta(v - mu/3) + \gamma(v^2 - 14muv/3 + m^2u^2/9)],$$
$$P = (p_1 - mp_2/3)(p_1^2 + 10mp_1p_2/3 + m^2p_2^2/9), \quad w = -m(v + mu/3), \tag{4.13}$$

- Case 9

$$U = \alpha v^{-1/2} + \beta uv^{-1/2} + \gamma u,$$
$$P = 3p_1^2p_2, \quad w = -v, \tag{4.14}$$

- Case 10

$$U = \alpha(v - \rho u/3) + \beta u^{-1/2} + \gamma u^{-1/2}(v - \rho u),$$
$$P = 3p_1^2p_2 + \rho p_2^3, \quad w = u. \tag{4.15}$$

In the following we show that the Killing tensors that define the leading terms of the first integrals above are isometrically different. For the corresponding coefficients of those Killing tensors, see Table 1, we plan to employ the invariant theory of Killing tensors of valence three defined on the Minkowski plane, including the techniques based

Case #	Killing tensor	Non-vanishing coefficients
Case 1	$(up_1 - vp_2)^3$	$a_{10} = 1$
Case 2	$3(up_1 - vp_2)^2(p_1 + \mu p_2)$	$a_7 = 1, a_8 = -\mu$
Case 3	$3(up_1 - vp_2)^2(p_1^2 - a^2 p_2^2)$	$a_5 = 1, a_6 = a^2$
Case 4	$3p_2[(up_1 - vp_2)^2 - a^2 p_1^2]$	$a_3 = a^2, a_8 = -1$
Case 5	$3p_1 p_2(up_1 - vp_2)$	$a_9 = -1$
Case 6	$3p_2^2(up_1 - vp_2)$	$a_6 = -1$
Case 7	$(up_1 + 3mvp_2)^2(p_1 - 3mp_2)$	$a_1 = -1, \quad a_2 = 27m^3,$ $a_3 = -3m, \quad a_4 = 9m^2$
Case 8	$(p_1 - \frac{m}{3}p_2)(p_1^2 + \frac{10}{3}mp_1p_2 + \frac{m^2}{9}p_2^2)$	$a_1 = -1, \quad a_2 = \frac{m^3}{27},$ $a_3 = -m, \quad a_4 = m^2$
Case 9	$3p_1^2 p_2$	$a_3 = -1$
Case 10	$3p_1^2 p_2 + \rho p_2^3$	$a_2 = -\rho, a_4 = -1$

Table 1: Drach's Killing tensors cubic in the momenta

on invariants and covariants as well as the analysis based on the dimensions of the corresponding orbits.

We first note that in view of the ten Killing tensors (refer to (4.5)) corresponding to the ten cases listed by Drach, the action of the isometry group $I(\mathbb{E}_1^2)$ on the vector space $\mathcal{K}_0^3(\mathbb{E}_1^2)$ is not semi-regular everywhere. Indeed, it is straightforward to check the rank of the following matrix (with respect to the basis ∂_{a_i}, $i = 1, \ldots, 10$) resulting from the three infinitesimal generators at each point,

$$\begin{pmatrix} -3a_5 & 0 & a_9 & a_6 & 2a_7 & 0 & a_{10} & 0 & 2a_8 & 0 \\ 0 & -3a_6 & a_5 & -a_9 & 0 & -2a_8 & 0 & a_{10} & 2a_7 & 0 \\ -3a_1 & 3a_2 & -a_3 & a_4 & -2a_5 & 2a_6 & -a_7 & a_8 & 0 & 0 \end{pmatrix}. \tag{4.16}$$

As a result, we obtain the rank of each orbits containing the corresponding Killing tensors above respectively. See Table 2 for more details. Thus, for example, one cannot simply use the fundamental invariants to solve the classification problem. A more subtle scheme is required.

We begin by considering Case 1 (refer to Table 2). Since a_{10} is an invariant of the full isometry group $I(\mathbb{E}_1^2)$, it can be used to distinguish Case 1 from the rest. Observe next that Case 1 and Case 5 are the only cases where the corresponding orbits of group action are two-dimensional, this immediately distinguishes Case 5 from the remaining eight cases. Indeed, the invariant a_{10}) distinguishes the Killing tensor of Case 5 from that of Case 1 and the fact that its orbit is two-dimensional shows that it is isometrically distinct from the rest.

Since Cases 2,3,4 and 6 belong to three-dimensional orbits, while Case 7-10 have one-dimensional orbits, these two groups are immediately distinguished. It remains to

Case #	Rank of generators	Non-vanishing generators
Case 1	2	$\mathbf{U}_1, \mathbf{U}_2$
Case 2	3	$\mathbf{U}_1, \mathbf{U}_2, \mathbf{V}$
Case 3	3	$\mathbf{U}_1, \mathbf{U}_2, \mathbf{V}$
Case 4	3	$\mathbf{U}_1, \mathbf{U}_2, \mathbf{V}$
Case 5	2	$\mathbf{U}_1, \mathbf{U}_2$
Case 6	3	$\mathbf{U}_1, \mathbf{U}_2, \mathbf{V}$
Case 7	1	$\mathbf{V}$
Case 8	1	$\mathbf{V}$
Case 9	1	$\mathbf{V}$
Case 10	1	$\mathbf{V}$

Table 2: Rank analysis of the infinitesimal generators

distinguish the four cases in the first group and the four cases in the second group.

For Cases 2,3,4 and 6, one can use invariant $\Delta_2 = a_9 a_{10} - 2a_7 a_8$ to distinguish Case 2 from the other three, since it does not vanish for Case 2 while it does for Case 3, 4 and 6. Now consider the invariant submanifold defined by

$$S = \{a_7 = a_8 = a_{10} = 0\}.$$

One employs the method of infinitesimal generators to find one of the reduced invariants is

$$\Delta_8 = a_5 a_6, \tag{4.17}$$

which can be used to distinguish Case 3 from Case 4 and 6. To distinguish Case 4 from Case 6, one computes the fundamental covariants (see [12]).

Case 4. $\Delta_3^C = 2uv(a^2 - u^2),\ \Delta_3^C = 3v^2(1 + u^2),\ \Delta_i^C = 0, i \neq 3, 7,$

Case 6. $\Delta_i^C = 0, i = 1, \ldots, 9.$

This distinguishes Case 4 from Case 6.

Finally, for the last 4 cases, note the only non-vanishing generator is $\mathbf{V}$, which means that in the subspace where these four Killing tensors are located (i.e., the subspace characterized by the condition $a_i = 0,\ i \neq 1, 2, 3, 4$) the subgroup of translation is in fact an isotropy subgroup. The only transformations that matter are the rotations. This will distinguish Cases 7-10 from each other. The classification is now complete.

5. Concluding remarks

It is shown, from a the viewpoint of isometry group invariants, that the ten potentials of Drach are in fact distinct, Our next goal is to determine whether or not the Drach list is exhaustive. If not, that is, there are other integrable systems of Drach type, we wish to determine whether or not they are superintegrable.

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