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Minimum Cycle Bases of The Semi-Strong Product of Some Basic Graphs

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Abstract

The the cycle space, $\mathcal{C}(G)$, of a given graph has many bases. We are interested in finding a basis with minimum length called a minimum cycle basis. The length of a cycle basis is the sum of all the lengths of its cycles, where the length of a cycle is the number of edges in it.

In this paper we prove that the cycle spaces of the Semi-Strong product of paths with paths, paths with cycles, cycles with paths and cycle with cycle have minimum cycle bases consisting of 4-cycles. Also we give the lengths of these bases.

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1. Introduction

Throughout this paper we consider only finite, connected, simple, unweighted, undirected graphs. For the undefined terms see [3]. Let G be a graph, the set W_E of all subsets of the edge set of G, $E(G)$, form an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X \oplus Y = (X \backslash Y) \cup (Y \backslash X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for all $X, Y \in W_E$ ($\oplus$ is the ring sum). A cycle space, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(W_E, \oplus, \cdot)$ spanned by the cycles of G. It follows that the non-zero elements of $\mathcal{C}(G)$ are cycles or union of edge disjoint cycles. A *cycle basis* of G is a basis of $\mathcal{C}(G)$. *The dimension of the cycle space* denoted $\dim \mathcal{C}(G)$ is $|E(G)| - |V(G)| + 1$. The length of a cycle C is the number of its edges and denoted $|C|$. The length of a cycle basis $\mathcal{B}$, denoted $\rho(\mathcal{B})$, is the the sum of the lengths of all cycles in $\mathcal{B}$. A *minimum cycle basis* is a cycle basis of minimum length. In general, the minimum cycle bases are not very well behaved under graph operations in the sense that there is no way to extend these properties from subgraphs to the whole graph.

The cycle space and its analysis has various applications in different fields of sciences like electrical engineering [5], structural analysis [1], biology and chemistry [2] and periodic timetabling [6]. Some of these applications require cycle bases with special properties [7].

Graph products are the best natural way to enlarge the space of graphs. In the literature, one can find a lot of graph products such as; Cartesian product, direct product, strong product, semi strong bproduct and lexicographic product and many more products. Imrich and Stadler [4] found minimum cycle bases of product graphs; in fact they constructed minimum cycle bases of the Cartesian and strong products of graphs, they were interested in these two products being they are commutative products.

Definition 1.1. The Semi-Strong product of two graphs G_1 and G_2, denoted by $G_1 * G_2$, is a graph whose vertex set is $V(G_1 * G_2) = V(G_1) \times V(G_2)$ and edge set is

$$E(G_1 * G_2) = \left\{ \begin{array}{c} (u_1, v_1)\,(u_2, v_2) : \text{ either } [u_1u_2 \in E_1 \text{ and } v_1 = v_2] \\ \text{or } [u_1u_2 \in E_1 \text{ and } v_1v_2 \in E_2] \end{array} \right\}.$$

It is clear that $|E(G_1 * G_2)| = 2\,|E(G_1)|\,|E(G_2)| + |V(G_2)|\,|E(G_1)|$ and $|V(G_1 * G_2)| = |V(G_1)|\,|V(G_2)|$, thus $\dim\mathcal{C}\,(G_1 * G_2) = |E(G_1 * G_2)| - |V(G_1 * G_2)| + 1$.

The purpose of this paper is to construct minimum cycle bases of the semi-strong product of paths, cycles and paths with cycles with emphasizing on the fact that the semi-strong product is neither noncommutative nor associative product and doesnot behave well, so we study the semi-strong product of graphs on their own.

2. The Main Results

In this section, we use the notations P_n, C_n to denote a path and a cycle respectively. We will investigate the minimum cycle bases of the semi strong product of the graphs $P_n * P_m$, $P_n * C_m$, $C_n * P_m$ and $C_n * C_m$. Throughout this work, we use P_n to denote the path $1, 2, \ldots, n$ and C_n to denote the cycle $1, 2, \ldots, n1$, such that $E(P_n) = \{i(i+1) : 1 \leqslant i \leqslant n-1\}$ and $E(C_n) = E(P_n) \cup \{n1\}$.

Note that $|V(P_n * P_m)| = mn$, $|E(P_n * P_m)| = 2(n-1)(m-1) + m(n-1) = 3mn - 2n - 3m + 2$, and so $\dim \mathcal{C}\,(P_n * P_m) = 3mn - 2n - 3m + 3$.

Theorem 2.1. The cycle spsce of $P_n * P_m$ has minimum cycle basis consisting of 4-cycles with length $8mn - 6m - 8n + 12$.

Proof. Our object is to construct a cycle basis of minimum length for $\mathcal{C}(P_n * P_m)$. For each $i = 1, 2, \ldots, n-1$ we define the set of cycles

$$A_i = \{(i, j)\,(i+1, j)\,(i, j+1)\,(i+1, j+1)\,(i, j) : 1 \leq j \leq m-1\},$$

also, for each $i = 1, 2, \ldots, n-2$, we define the set of cycles

$$\begin{aligned} B_i &= \{(i, j)\,(i+1, j-1)\,(i+2, j)\,(i+1, j+1)\,(i, j) : 2 \le j \le m-1\} \\ &\quad \cup \{(i, 1)\,(i+1, 1)\,(i+2, 1)\,(i+1, 2)\,(i, 1)\}, \end{aligned}$$

then, we set $A = \bigcup_{i=1}^{n-1} A_i$ and $B = \bigcup_{i=1}^{n-2} B_i$. It is clear that, $|A| = (n-1)\,(m-1)$ and $|B| = (n-2)\,(m-1)$. If we define $\mathcal{B}\,(P_n * P_m) = A \cup B$, then

$$|\mathcal{B}\,(P_n * P_m)| = 2mn - 3m - 2n + 3 = \dim \mathcal{C}(P_n * P_m).$$

Since $|\mathcal{B}\,(P_n * P_m)| = \dim \mathcal{C}(P_n * P_m)$, it is enough to prove that $\mathcal{B}\,(P_n * P_m)$ is linearly independent set of cycles to guarantee that it is a basis of $\mathcal{C}(P_n * P_m)$.

Note that each $i(i+1) * P_m$ is isomorphic to the graph $i(i+1) \times P_m$, and so it is planar and has A_i as the set of all boundaries of its finite faces, thus A_i is a basis of $\mathcal{C}(i\,(i, i+1) * P_m)$, for each $i = 1, 2, \ldots, n-1$. Since each A_i is linearly independent and $E(A_i) \cap E(A_j) = \varphi$, for all $i \neq j$ the set $A = \bigcup_{i=1}^{n-1} A_i$ is linearly a independent set of cycles in $\mathcal{C}(P_n * P_m)$.

For each $i = 1, 2, \ldots, n-2$, B_i contains edge disjoint cycles, so the cycles of B_i are linearly independent, thus B_i is a linearly independent set of cycles in $\mathcal{C}(P_n * P_m)$. Note that, $B_i \subset \mathcal{C}(i(i+1)\,(i+2) * P_m)$ and $B_{i+1} \subset \mathcal{C}((i+1)\,(i+2)\,(i+3) * P_m)$. It is clear that each B_i contains one or two edges of the form $(i, j)\ (i+1, j-1)$ or $(i, j)\,(i+1, j+1)$ which donot occur in any other cycle of $B_i \cup B_{i+1}$, so such a cycle cannot be obtained from any linear combination of cycles of $B_i \cup B_{i+1}$ which implies that all cycles of $B_i \cup B_{i+1}$ are linearly independent and one can easily notice that $B_i \cap B_{i+2} = \varphi$ for all $i = 1, 2, \ldots, n-4$. Then, we conclude that $B = \bigcup_{i=1}^{n-2} B_i$ is a linearly independent set of cycles in $\mathcal{C}(P_n * P_m)$.

Every linear combination of cycles from A contains one or two edges of the form (i, j)

$(i+1, j)$; $j = 1, 2, \ldots, m$ and $i = 2, 3, \ldots, n-1$ which doesnot occur in any cycle of B, thus all linear combinations of cycles of A are linearly independent with B, then $A \cup B = \mathcal{B}\,(P_n * P_m)$ is linearly independent set of cycles in $\mathcal{C}(P_n * P_m)$. Therefore, $\mathcal{B}\,(P_n * P_m)$ is a basis of $\mathcal{C}(P_n * P_m)$. Moreover, it is clear, from the construction of $\mathcal{B}(P_n * P_m)$, that all cycles in $\mathcal{B}(P_n * P_m)$ are 4-cycles, and so, $\rho\,(\mathcal{B}(P_n * P_m)) = 8mn - m - 8n + 12$. Since the girth of $P_n * P_m$ is 4, $\mathcal{B}(P_n * P_m)$ is a minimum cycle basis of $\mathcal{C}(P_n * P_m)$. ■

Now, we consider the graph $P_n * C_m$ which has $|V(P_n * C_m)| = mn$ and $|E(P_n * C_m)| = 3mn - 3m$, and so $\dim \mathcal{C}(P_n * C_m) = 2mn - 3m + 1$.

Theorem 2.2. The cycle spsce of $P_n * C_m$ has minimum cycle basis consisting of 4-cycles with length $8mn - 12m + 4$.

Proof. It is clear that $P_n * C_m$ can be obtained from $P_n * P_m$ by adding the set of edges $D = \{(i, 1)\,(i + 1, m)\,, (i, m)\,(i + 1, 1) : 1 \leq i \leq n - 1\}$, $|D| = 2(n - 1)$. So, all what we need is to find $2(n - 1)$ linearly independent cycles in $\mathcal{C}(P_n * C_m)$ that are linearly independent with all cycles of $\mathcal{B}(P_n * P_m)$. Define the set of cycles $\mathcal{B}(P_n * C_m) = \mathcal{B}(P_n * P_m) \cup K \cup L$ where

$$\begin{aligned} K &= \{(i, 1)\,(i + 1, m)\,(i, m)\,(i + 1, 1)\,(i, 1) : 1 \leq i \leq n - 1\} \\ L &= \{(i, 1)\,(i + 1, m)\,(i + 2, 1)\,(i + 1, 2)\,(i, 1) : 1 \leq i \leq n - 2\} \cup \\ &\quad \{a = (1, m)\,(2, 1)\,(3, m)\,(2, m - 1)\,(1, m)\}\,. \end{aligned}$$

The cycles of K are linearly independent because they are edge disjoint. The cycles of $L\backslash\{a\}$ are edge disjoint, so they are linearly independent and each of these cycles contains two edges of the form $(i, 1)\,(i + 1, 1)$ and $(i, 2)\,(i + 1, 1)$ which donot occur in any cycle of K, thus the cycles of $L\backslash\{a\}$ are linearly independent with all the cycles of K. The cycle a has the two edges $(3, m)\,(2, m - 1)$ and $(2, m - 1)\,(1, m)$ which dont occur in any cycle of $(K \cup L) \setminus \{a\}$, so it is linearly independent with all these cycles. Therefore, $K \cup L$ is linearly independent set of cycles in $\mathcal{C}(P_n * C_m)$. Moreover, every linear combination of cycles from $K \cup L$ contains two edges from the set D which donot occur in any cycle of $\mathcal{B}(P_n * P_m)$, so all cycles of $K \cup L$ are linearly independent with all cycles of $\mathcal{B}(P_n * P_m)$, hence $\mathcal{B}(P_n * C_m) = \mathcal{B}(P_n * P_m) \cup K \cup L$ is linearly independent set of cycles in $\mathcal{C}(P_n * C_m)$ and since

$$\begin{aligned} |\mathcal{B}(P_n * C_m)| &= |\mathcal{B}(P_n * P_m)| + |K| + |L| \\ &= (2mn - 3m - 2n + 3) + (n - 1) + (n - 1) \\ &= 2mn - 3m + 1 = \dim \mathcal{C}(P_n * C_m), \end{aligned}$$

we conclude that $\mathcal{B}(P_n * C_m)$ is a basis of $\mathcal{C}(P_n * C_m)$. Since the girth of $P_n * C_m$ is 4 and $\mathcal{B}(P_n * C_m)$ contains only 4-cycles it must be a minimum cycle basis of $\mathcal{C}(P_n * C_m)$ and its length is $\rho\,(\mathcal{B}(P_n * C_m)) = 8mn - 12m + 4$. ■

We consider now the graph $C_n * P_m$ which can be obtained from the graph of $P_n * P_m$ by adding the set of edges

$$F = \{(n, j)\,(1, j + 1)\,, (n, j + 1)\,(1, j) : 1 \leq j \leq m - 1\} \cup \{(n, j)\,(1, j) : 1 \leq j \leq m\}$$

It is clear that $|F| = 3m - 2$, $|V(C_n * P_m)| = mn$ and $|E(C_n * P_m)| = 3mn - 2n$. Thus, $\dim \mathcal{C}(C_n * P_m) = 2mn - 2n + 1$.

Theorem 2.3. The cycle spsce of $C_n * P_m$ has a minimum cycle basis all of its cycles are of length 4 except one of length n and $\rho\,(C_n * P_m) = 8n\,(m - 1) + n$.

Proof. We define the set of cycles $\mathcal{B}(C_n * P_m) = \mathcal{B}(P_n * P_m) \cup B_{n-1} \cup B_n \cup A_n \cup \{Q\}$, where B_{n-1}, B_n, A_n and Q are defined as follows:

$$\begin{aligned} B_{n-1} &= \{(n-1, j)\,(n, j-1)\,(1, j)\,(n, j+1)\,(n-1, j) : 2 \le j \le m-1\} \cup \\ &\quad \{(n-1, 1)\,(n, 1)\,(1, 1)\,(n, 2)\,(n-1, 1)\} \\ B_n &= \{(n, j)\,(1, j-1)\,(2, j)\,(1, j+1)\,(n, j) : 2 \le j \le m-1\} \cup \\ &\quad \{(n, 1)\,(1, 1)\,(2, 1)\,(1, 2)\,(n, 1)\} \\ A_n &= \{(n, j)\,(1, j)\,(n, j+1)\,(1, j+1)\,(n, j) : 1 \le j \le m-1\} \\ Q &= (1, m)\,(2, m) \cdots (n, m)\,(1, m)\,. \end{aligned}$$

It is clear that B_{n-1}, B_n and A_n are defined the same way we have defined the sets B_i 's and A_i 's in Theorem 2.1 just by replacing i by n or $n-1$. Thus, we can redefine $\mathcal{B}(C_n * P_m)$ as follows:

$$\mathcal{B}(C_n * P_m) = \left(\bigcup_{i=1}^{n} A_i\right) \cup \left(\bigcup_{i=1}^{n} B_i\right) \cup \{Q\}\,.$$

If we mimic the proof of Theorem 2.1 we conclude that $\left(\bigcup_{i=1}^{n} A_i\right) \cup \left(\bigcup_{i=1}^{n} B_i\right)$ is linearly independent set of cycles in $\mathcal{B}(C_n * P_m)$. We notice that Q has no edges in common with all the cycles in $\left(\bigcup_{i=1}^{n} B_i\right)$ and it has one edge in common with the cycles of A_i for each i, so Q cannot be obtained from any linear combination of cycles from $\mathcal{B}(C_n * P_m) \setminus \{Q\}$. Hence, $\mathcal{B}(C_n * P_m)$ is a linearly independent set of cycles in $\mathcal{C}(C_n * P_m)$ and one can easily see that $|\mathcal{B}(C_n * P_m)| = 2mn - 2n + 1 = \dim \mathcal{C}(C_n * P_m)$. Therefore, $\mathcal{B}(C_n * P_m)$ is a basis of $\mathcal{C}(C_n * P_m)$, moreover, it is of minimum length being all of its cycles are of length equals to the girth of $C_n * P_m$ except one of length n, and so, $\rho\,(C_n * P_m) = 8n\,(m-1) + n$.

Now, we consider the graph $C_n * C_m$ which may be recognized as a graph obtained from the graph $C_n * P_m$ by adding the set of edges

$$T = D \cup N;\, N = \{(n, 1)\,(1, m)\,,\,(n, m)\,(1, 1)\}\,,\ |T| = 2n$$

where D is defined before Theorem 2.2. It is clear that $|V(C_n * C_m)| = mn$ and $|E(C_n * C_m)| = 3mn$, so $\dim \mathcal{C}(C_n * C_m) = 2mn + 1$. ■

Theorem 2.4. The cycle spsce of $C_n * C_m$ has a minimum cycle basis all of its cycles are of length 4 except one of length n and $\rho\,(C_n * P_m) = 8mn + n$.

Proof. We define the set of cycles $\mathcal{B}(C_n * C_m) = \mathcal{B}(C_n * P_m) \cup K \cup L \cup \{k, l\}$; where k and l are two cycles defined, the same way we have defined the cycles of K and L, as follows:

$$k = (n, 1)\,(1, 1)\,(n, m)\,(1, m)\,(n, 1) \text{ and } k = (n-1, 1)\,(n, m)\,(1, 1)\,(n, 2)\,(n-1, 1)\,.$$

Using similar arguements to those used in the proof of Theorem 2.2 one can easily verify that $\mathcal{B}(C_n * P_m) \cup K \cup L$ is linearly independent set of cycles being $K \cup L$ has no common edges with the cycles of B_{n-1}, B_n and A_n. It is easy to notice that k and l are linearly independent and each of them contains one or two of the edges (n, m) $(1, 1)$ and $(1, m)$ $(n, 1)$ which donot belong to any linear combination of cycles of $\mathcal{B}(C_n * C_m) \backslash \{k, l\}$, hence, $\mathcal{B}(C_n * C_m)$ is linearly independent set of cycles in $\mathcal{C}(C_n * C_m)$, and $|\mathcal{B}(C_n * C_m)| = 2mn + 1 = \dim \mathcal{C}(C_n * C_m)$. Therefore, $\mathcal{B}(C_n * C_m)$ is a basis of $\mathcal{C}(C_n * C_m)$ of minimum length because $C_n * C_m$ has girth 4 and all cycles of $\mathcal{B}(C_n * C_m)$ are 4-cycles excepe one of length n, and so $\rho\,(C_n * P_m) = 8mn + n$. ■

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