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# An Inventory Model with Shortage, Time-Depranent Demand Rate and Quantity Dependent Permissible Delay in Payment

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**Abstract**

In this paper authors studied an inventory model in which the permissible delay in payment depends on the order quantity and Time- Dependent demand rate. Numerical examples have been given to illustrate the model.

**Keywords:** Inventory, variable demand rate, shortage, Backlogging, Permissible delay in payments.

## 1. Introduction

In today's business transaction, the customer does not have to pay any interest during the fixed period. This fixed period is called grace period. But if the payment gets delayed beyond the grace period, interest will be charged by the supplier or the producer. This arrangement comes out to be very advantageous to the customer as he may delay the payment till the end of the permissible delay period. Thus, it makes economic sense for the customer to delay the payment of the replenishment account up to the last day of the settlement period allowed by the supplier. Manisha Pal and S.K Ghose [7] studied an inventory model with shortage and quantity dependent permissible delay in payment. Goyal [4] developed an economic order quantity (EOQ) model under the condition of permissible delay in payments first. Shinn *et al.* [1] extended the model by considering quantity discount for freight cost. Aggarwal and Jaggi [3] and Hwang and Shinn [1] recently extended Goyal's model to consider deterministic inventory model with constant rate of deterioration. Shah and Shah [2] developed probabilistic inventory model for deteriorating items when delay in payment is permitted. Jamal [6] extended Aggarwal and Jaggi's model to allow for shortage.

The present paper incorporate this fact in an inventory model allowing shortage with variable demand rate and obtain the optimal ordering policy. In the section 2 assumptions and notations are presented. In section 3 the mathematical model is formulated and some results are proved .At last in section 4 numerical examples is given to illustrate the model.

## 2. Assumptions and Notations

The following notations and assumptions are used in this paper to develop the proposed model.

### 2.1 Notations

K = Ordering cost of inventory per order
P = per unit purchase cost
s = per unit shortage cost
h = per unit holding cost excluding interest changes
$I_e$ = Interest which can be earned
$I_r$ = Interest charges which invested in inventory, $I_r \geq I_e$
T = Length of replenishment cycle
$T_1$ = Time taken inventory level comes down to zero $0 \leq T_1 < T$
l(t) = Inventory level at time t
$Z_M(T_1, T)$ = average total inventory cost per unit time when permissible delay period in payment is M.

Let $$Z_M(T_1, T) = \begin{cases} Z_M^1(T_1, T) & \text{for } T \geq M \\ Z_M^2(T_1, T) & \text{for } T < M \end{cases}$$

### 2.2 Assumptions

1. The inventory system involves only one item.
2. Demand ratio R(t) is deterministic and given by
   $R(t) = \alpha t;\ 0 < t < T.$

3. Replenishment occurs instantaneously on ordering i.e. lead-time is zero.
4. Shortage are allowed and completely back logged.
5. The planning period is of infinite length. The planning horizon is divided into sub-intervals of length T units. Order is placed at time points, 0, T, 2T, 3T ,.... the order quantity at each recorder point being just sufficient to bring the stock height to a certain maximum level S.
6. The length of the permissible delay period M for repaying the supplier is given by

$M = M_1$ , if $0<q<q_0$ and $M = M_2$ , if $q> q_0$ .
where q is the ordered quantity and $q_0$ a specified value of q and $M_2 > M_1$ .

No payment to the supplier is outstanding at the time of placing an order i.e. M. < T.

## 3. Model Formulation

Since the planning period is of infinite length, we study the model over a recorder interval; say (0, T). Two situations can arise, which are described pictorially in figure 1 and 2. Variation of inventory level I(t) with respect of time is given by

$$\frac{dI(t)}{dt} = -\alpha t \,, \quad 0 < t < T \qquad (1)$$

The solution of equation (1) is given by

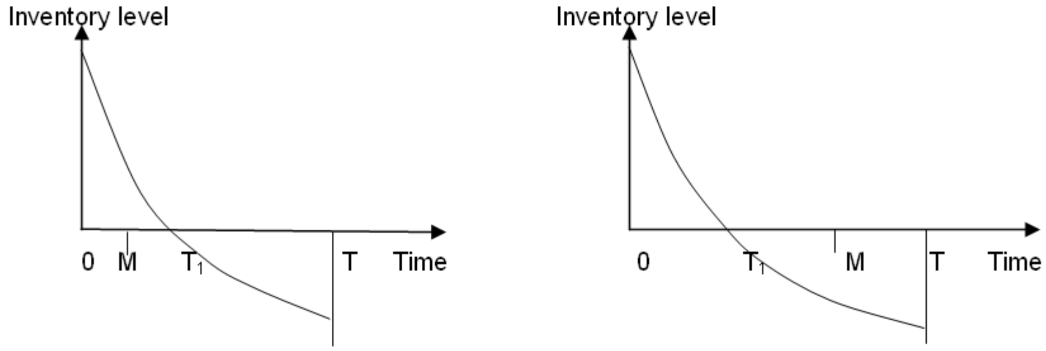
$$I(t) = \frac{\alpha}{2}\left(T_1^2 - t^2\right), \; 0 < t < T \qquad (2)$$

with boundary condition $I(T_1) = 0$
In the interval (0, T)

$$\text{Expected Holding Cost HC} = h\int_0^{T_1} I(t)dt = \frac{\alpha h T_1^3}{3} \qquad (3)$$

$$\text{Expected Shortage Cost SC} = s\int_{T_1}^{T} I(t)dt$$

$$\therefore \text{SC} = \frac{s\alpha}{6}\left(3T_1^2 T - 3T_1^3 T - T^3\right) \qquad (4)$$




Case 1: $M \leq T_1$ Case 2 : $M > T_1$

**Case 1: $M \leq T_1$**

In this situation since the length of period with positive stock is larger than the credit period, the buyer can use the sale revenue to earn interest at an annual rate $I_c$ in the interval (O, $T_1$). The interest earned $IE_1$ is given by :

$$IE_1 = PI_e \int_0^T \frac{\alpha}{2}\left(T_1^2 - t^2\right)dt \;\; = \frac{\alpha\, PI_e T_1^3}{3} \tag{5}$$

The unsold stock is supposed to be financed with an annual rate $I_r$ beyond the credit limit. The interest; payable IP is given by

$$IP = PI_r \int_M^{T_1} \frac{\alpha}{2}\left(T_1^2 - t^2\right)dt \;\; = \frac{\alpha\, PI_r}{6}\left(2T_1^3 - 3T_1^2 M + M^3\right) \tag{6}$$

Therefore the total average cost per unit time is given by

$$Z_M^1(T_1, T) = \frac{[K + HC + SC + IP - IE_1]}{T}$$

$$= \frac{1}{T}\left[K + \frac{s\alpha}{6}\left(3T_1^2 T - 3T_1^3 T - T^3\right) + \frac{\alpha\, PI_r}{6}\left(2T_1^3 - 3T_1^2 M + M^3\right) - \frac{\alpha\, PI_e T_1^3}{3}\right] \tag{7}$$

Optimal value of $T_1$ and T which maximize $Z_M^1(T_1, T)$ are obtained by solving the equations

$\frac{\partial Z_M^1(T_1, T)}{\partial T_1} = 0$ and $\frac{\partial Z_M^1(T_1, T)}{\partial T} = 0$, which give

$$[h - s + P\,(I_r - I_e)]T_1 + sT = PI_r M \tag{8}$$

and

$$\frac{1}{T^2}\left[K + \frac{\alpha h T_1^3}{3} + \frac{\alpha s}{6}\left(3T_1^2 - 2T_1^3 - T^3\right) + \frac{\alpha PI_r}{6}\left(2T_1^3 - 3T_1^2 M + M^3\right) - \frac{\alpha PI_e T_1^3}{3}\right]$$

$$+ \frac{1}{T}\frac{\alpha s}{2}\left(T^2 - T_1^2\right) = 0 \tag{9}$$

**Case 1: M > $T_1$**

The buyer pays no interest but earns at an annual rate $I_e$ during the period (0, M) because M > $T_1$. Interest earned in this case, denoted by $IE_2$ is given by.

$$IE_2 = PI_e\left[\int_0^{T_1} \alpha\, t^2 dt + (M - T_1)\int_0^{T_1} \alpha\, t dt\right] \;\; = \frac{PI_e \alpha\, T_1^2}{2}\left(M - \frac{T_1}{3}\right) \tag{10}$$

Then the total average cost per unit time is

$$Z_M^2(T_1, T) = \frac{[K + HC + SC - IE_2]}{T}$$

$$= \frac{1}{T}\left[K + \frac{\alpha\, s}{2}\left(3T_1^2 T - 2T_1^3 - T^3\right) + \frac{\alpha\, hT_1^3}{3} - \frac{PI_e\alpha\, T_1^2}{2}\left(M - \frac{T_1}{3}\right)\right] \quad (11)$$

Optimal values of $T_1$ and T which minimize $Z_M^2(T_1, T)$ are obtained by solving the equations

$\frac{\partial Z_M^2(T_1, T)}{\partial T_1}$ = 0 and $\frac{\partial Z_M^2(T_1, T)}{\partial T}$ = 0, which give

$$\left(h - s + \frac{PI_e}{2}\right)T_1 = PI_eM - sT \quad (12)$$

and

$$\frac{1}{T}\left[\frac{\alpha s}{2}(T_1 - T)(T_1 + T)\right] - \frac{1}{T^2}\left[K + \frac{\alpha s}{6}\left(3T_1^2 T - 2T_1^3 - T^3\right) + \frac{\alpha hT_1^3}{3} - \frac{PI_e\alpha T^2}{2}\left(M - \frac{T_1}{3}\right)\right] = 0 \quad (13)$$

**Theorem 1:** Optimal $T_1$ is an increasing function of T, if $sT > PI_rM$.

**Proof:** From equation (8) we have

$$T_{1opt} = \frac{sT - PI_rM}{\left[s - \{h + P(I_r - I_e)\}\right]}$$

$$\frac{dT_{1opt}}{dT} = \frac{s}{s - \{h + P(I_r - I_e)\}} = C; \text{ (say)}$$

which is constant and independent of T, since $I_r > I_e$ then $C > 1$.

From equation (12) we have

$$T_{1\ opt}^{*} = \frac{sT - PI_eM}{s - h + \frac{PI_e}{2}} \quad (14)$$

$$\frac{dT_{1\,opt}^{*}}{dT} = \frac{s}{s - \left(h + \frac{PI_e}{2}\right)} = b; \text{ (say)}$$

$\frac{dT^*_{1opt}}{dT}$ is constant and independent of T clearly b>1. Hence optimal $T_1$ is an increasing function of T.

**Theorem 2 :** Optimal T is an increasing function of M, if sT > $PI_eM$.

**Proof:** Substituting $T_{1opt}$ from equation (13) in $Z^1_M(T_1, T)$, we get

min $Z^1_M(T_1, T)$ = $Z^1_M$(T), say optimal T, is obtained by solving $\frac{\partial}{\partial T} Z^1_M(T)$ =0, which gives

$$\frac{-N}{T^2} + \frac{\alpha}{T}\left[\left\{hC - \frac{s}{2}(2C-1) + PC(I_r - I_e)\right\}T^2_{1opt} + C(sT - PI_rM)T_{1opt} - \frac{s}{2}T^2\right] = 0$$

where N is the numerator of $Z^1_M$(T). Hence

$$\frac{N}{T} = \alpha\left[\left\{hC - \frac{s}{2}(2C-1) + PC(I_r - I_e)\right\}T^2_{1opt} + C(sT - PI_rM)T_{1opt} - \frac{s}{2}T^2\right] \quad (15)$$

Differentiating equation (15) w.r.t. M we get

$$\frac{\partial\left(\frac{N}{T}\right)}{\partial M} = \frac{\partial\left(\frac{N}{T}\right)}{\partial M}\frac{\partial T}{\partial M} = \alpha\begin{bmatrix}\left\{hC - \frac{s}{2}(2C-1) + PC(I_r - I_e)\right\}2T_{1opt}\ x \\ C^2\left(\frac{\partial T}{\partial M} - \frac{PI_r}{s}\right) + C\left(s\frac{\partial T}{\partial M} - pI_r\right)T_{1opt} + (sT - PI_rM)C^2\left(\frac{\partial T}{\partial M} - \frac{PI_r}{s}\right) - sT\frac{\partial T}{\partial M}\end{bmatrix}$$

$\frac{\partial\left(\frac{N}{T}\right)}{\partial M}\frac{\partial T}{\partial M} = 0$ gives

$$\frac{\partial T}{\partial M} = \frac{CPI_r}{s}\frac{\left[(s+2a)T_{1opt} + C(sT - PI_rM)\right]}{\left[C(s+2a)T_{1opt} + C^2(sT - PI_rM) - sT\right]} > 0$$

if sT > $PI_rM$ since C > 1 and $I_r > I_e$,where, hc – $\frac{s}{2}$(2C–1) + PC($I_r - I_e$) = a

Again, let $\min_{T_1} Z^2_M(T_1, T) = Z^2_M(T)$, say

Optimal T is obtained by solving $\frac{\partial}{\partial T} Z^2_M(T) = 0$ which gives

$$\frac{N}{T}=\alpha\left[\left(\frac{s}{2}+hb-sb-\frac{PI_e b}{2}\right)T_{1opt}^{*2}+b(sT-PI_eM)T_{1opt}^{*}-\frac{s}{2}T^2\right] \tag{16}$$

Differentiate (16) w.r.t. M we get

$$\frac{\partial\left(\frac{N}{T}\right)}{\partial T}\frac{\partial T}{\partial M}=\alpha\left[P^2bT_{1opt}^{*}\left(\frac{\partial T}{\partial M}-\frac{PI_e}{s}\right)+b\left(s\frac{\partial T}{\partial M}-pI_e\right)T_{1opt}^{*}+b^2(sT-PI_eM)\left(\frac{\partial T}{\partial M}-\frac{PI_r}{s}\right)-sT\frac{\partial T}{\partial M}\right]$$

$\frac{\partial\left(\frac{N}{T}\right)}{\partial T}\frac{\partial T}{\partial M}=0$ .gives

$$\frac{\partial T}{\partial M}=\frac{bPI_r}{s}\frac{\left[(s+2P)T_{1opt}^{*}+b(sT-PI_rM)\right]}{\left[b(s+2P)T_{1opt}^{*}+b^2(sT-PI_eM)-sT\right]}>0\text{ , if } sT>PI_eM$$

Hence the theorem.

**Theorem 3:** $Z_M^1$(T) is convex in T.

**Proof.** $\frac{\partial}{\partial T}Z_M^1(T)=0$, where N is the numerator of $Z_M^1(T)$

$$\frac{\partial^2 Z_M^1(T)}{\partial T^2}=2\frac{N}{T^3}-\frac{1}{T^2}\frac{dN}{dT}-\frac{1}{T^2}\frac{dN}{dT}+\frac{1}{T^2}\frac{d^2N}{dT^2}$$

$$=\frac{1}{T^2}\frac{d^2N}{dT^2}\quad\text{(since N = } T\frac{dN}{dT}\text{)}$$

$$=\frac{\alpha}{T}\left[\begin{array}{r}2C^2\left\{hC^2+s(1-C)+PC(I_r-I_c)\right\}\left(T-\frac{PI_rM}{s}\right)\\ -s(C^2-1)T-C^2PI_rM\end{array}\right]>0$$

as $I_r > I_e$ and hC > s (C-1). It is possible only if $Z_M^1(T)$ is convex in T.

## 4. Numerical Examples

Let K = Rs. 6.00 per order, h = Rs. 2.5 per unit, P = Rs. 70.00 per unit, s = Rs. 20.00 per unit, $\alpha$ = 300 per unit, $I_r$ = 0.15, $I_e$ = 0.10 and

$$M=\begin{cases}5 & 0<q<7000\\ 20 & q\geq 7000\end{cases}$$

**Step 1.** Consider M = 20 days , For $T_1 \geq 20$, $T_{opt}$ = 24.5 days, $T_{1opt}$ = 20 days

$Z_{20}^{1}(T_{1opt}, T_{opt})$ = Rs. 87.88

For $T_1 < 20$, $T_{opt}$ = 20 days, $T_{1opt}$ = 18.5 days

$Z_{20}^{2}(T_{1opt}, T_{opt})$ = Rs. 108.18

Hence optimal T and $T_1$ minimizing $Z_{20}(T_1,T)$ are T* = 24.5 days, $T_1^*$ =20 days and $Z_{20}(T_1^*, T^*)$ = Rs. 87.88. Since $\alpha T^* = 7350 > q_0 = 7000$. Thus T* = 24.5 days, $T_1^*$ = 20 days are optimal with minimum cost per day Rs. 87.88.

Again consider M = 5 days,For $T_1 \geq 5$, $T_{opt}$ = 8 days, $T_{1opt}$ = 7.7 days and $Z_{5}^{1}(T_{1opt}, T_{opt})$ = Rs.273.63.

For $T_1 > 5$, $T_{opt}$ = 7.7 days, $T_{1opt}$ = 7.25 days and $Z_{5}^{2}(T_{1opt}, T_{opt})$ =Rs.282.66, which verify theorem 4.

**Example 2.**
**Step 1.** Let K = Rs. 8 per order, h = Rs. 5 per unit, P = Rs. 100 unit, s = Rs. 25 per unit, $\alpha$ = 200 units, $I_r$ = 0.15, $I_e$ = 0.10 and

$$M = \begin{cases} 20 & 0 < q < 6000 \\ 35 & q \geq 6000 \end{cases}$$

Consider M = 35,For $T_1 \geq 35$, $T_{opt}$ = 31.67, $Z_{35}^{1}(T_{1opt}, T_{opt})$ = Rs. 69.99,$T_1 < 35$, $T_{opt}$ = 31.67, $T_{1opt}$ = 29.45, $Z_{35}^{2}(T_{1opt}, T_{opt})$ = Rs. 42.36.

Hence optimal T and $T_1$ minimizing, $Z_{35}(T_1, T)$ are T* = 31.67 and $T_1^*$ =29.45 days and minimum cost is $Z_{35}(T_1^*, T^*)$ = Rs. 42.36.

Since $\alpha T^* = 6334 > q_0 = 6000$, hence T* = 31.67 days, $T_1^*$ = 29.45 days are optimal with minimum cost per day = Rs. 42.36.

## 5. Conclusion

In this paper authors studied on inventory model where the shortage is completely backlogged and the permissible delay in payment depends on the order quantity with with demand rate depends upon time. An algorithm is suggested to find the optimal ordering policy, which helps the inventory manager to decide whether it would be a longer credit period for repaying the supplier by ordering a larger amount of the commodity. In this paper authors considered any one break in the delay period.

"Note that theorem 1. 2, and 3 here are a generalization of the corresponding results 1, 2, 3 and 4 of Manisha Pal (2006) in which demand rate is constant".

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