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A Multiquadric Collocation Method for Fredholm Integral Equations

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Abstract

In this paper we use the multiquadric radial basis function as a basis for a collocation method for Fredholm integral equations. The definite integral in the integral equation is approximated by a five point Gauss-Legendre quadrature method while the coefficients of the basis functions are obtained by solving the resulting linear system of equations. The efficiency of the method is demonstrated with two numerical examples.

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1. Introduction

In this paper we will solve numerically the linear Fredholm integral equation of the second kind given as

$$\lambda u(x) - \int_a^b K(x, y)u(y)\, dy = f(x) \tag{1.1}$$

with parameter λ. As shown in [2], this can be written in operator notation as

$$(\lambda - K)u = f \tag{1.2}$$

where K is a compact linear operator on a Banach space X with norm $\|\cdot\|$, and $f \in X$ is a given function and u is an unknown function belonging to X.

There are two main classes of numerical methods for solving (1.1): projection methods and Nyström methods. Collocation methods, which belong to the class of projection methods, are well established methods for differential equations, integro-differential equations and integral equations, see [1,5,7]. They generally provide accurate numerical approximations but their efficiency may depend on the choice of the basis functions and the collocation points.

Radial basis functions (RBFs) are powerful tools used in multivariate approximation theory and have been used extensively in the numerical solution of differential equations, mathematical finance and optimization amongst other areas. The efficiency of the RBF interpolation on scattered and uniform data has been illustrated by several numerical examples and theoretical results in Wendland [8] and Fasshauer [3].

In this paper, we will use the multiquadric, a type of radial basis function as a basis for a collocation method for Fredholm integral equations of the second kind. The choice of multiquadric is based on their good approximation properties and the fact that they have been used previously for collocation in partial differential equations by Kansa [4] and Lorentz [6].

2. Radial Basis Function Interpolation

Radial basis function interpolation to scattered data $(\mathbf{x}_k, \mathbf{u}_k \equiv \mathbf{u}(\mathbf{x}_k)) \in \mathbb{R}^{N+1}$ for pairwise distinct data sites $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^n$ uses a fixed radial function ϕ and the space $\mathcal{P}_m$ of polynomials in $\mathbb{R}^n$ of order less than or equal to m to define an interpolant in the form

$$s(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi(\|\mathbf{x} - \mathbf{x}_j\|) + \sum_{j=1}^{q} d_j p_j(\mathbf{x}), \tag{2.1}$$

through the linear system

$$\begin{aligned} &\sum_{j=1}^{N} c_j \phi(\|\mathbf{x}_i - \mathbf{x}_j\|) + \sum_{j=1}^{q} d_j p_j(\mathbf{x}_i), \qquad i = 1, \ldots, N \\ &\sum_{i=1}^{N} c_i p_j(\mathbf{x}_i) = 0, \qquad j = 1, \ldots, q \end{aligned} \tag{2.2}$$

where $p_1, \ldots, p_q$ is a basis for $\mathcal{P}_m$.

Possible choices for ϕ are, along with their order m, shown in Table 1.

Radial basis function interpolants have the nice property of being invariant under all Euclidean transformations (i.e. translations, rotations and reflections). This is because Euclidean transformations are characterized by orthogonal transformation matrices and are therefore Euclidean-norm-invariant [3].

When $m = 0$, which includes the multiquadrics, the interpolant s in (2.1) has the

Table 1: Radial basis functions (RBFs) and their orders.

RBF	$\phi(r)$	**Parameters**	**Order**
Polyharmonic Splines	r^{2k-d} for d odd $r^{2k-d}\log(r)$ for d even	$k \in \mathbb{N}, k > d/2$ $k \in \mathbb{N}, k > d/2$	k k
Gaussians	$\exp(-r^2)$		0
Multiquadrics	$(1+r^2)^\nu$	$\nu > 0, \nu \notin \mathbb{N}$	$\lceil \nu \rceil$
Inverse Multiquadrics	$(1+r^2)^\nu$	$\nu < 0$	0

form

$$s(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi(\|\mathbf{x} - \mathbf{x}_j\|). \tag{2.3}$$

The coefficients $\mathbf{c} = (c_1, \ldots, c_N)^T$ of s in (2.3) can be obtained by solving the linear system

$$A\mathbf{c} = \mathbf{u}, \tag{2.4}$$

where $A = ((\phi(\|\mathbf{x}_i - \mathbf{x}_j\|))_{1\le i,j\le N}$ and $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_N)^T$.

We note that in one space dimension, the multiquadric interpolant with $\nu = \dfrac{1}{2}$ is given as

$$s(x) = \sum_{j=1}^{n} c_j \sqrt{1 + |x - x_j|^2}. \tag{2.5}$$

This is the form we will use in this paper.

3. The Collocation Method

According to Atkinson and Han [2], a collocation method for (1.1) seeks a function $\hat{u}$ belonging to some function space which can be written as

$$\hat{u}(x) = \sum_{j=1}^{N} c_j \phi_j(x), \qquad x \in [a, b]. \tag{3.1}$$

This is substituted into (1.1), and the coefficients $c_1, \ldots, c_N$ are determined by making the equation to be exact in some sense. We will pick the distinct collocation points $x_1, \ldots, x_N \in [a, b]$ so that the $c_1, \ldots, c_N$ are determined by solving the linear system

$$\sum_{j=1}^{n} c_j \left\{ \lambda\phi_j(x_i) - \int_a^b K(x_i, y)\phi_j(y)\, dy \right\} = f(x_i), \qquad i = 1, \ldots, N. \tag{3.2}$$

We will follow the collocation method of Kansa [4] where the approximate solution $\hat{u}$ is represented by a radial basis function expansion analogous to that used for data interpolation, i.e.,

$$\hat{u}(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi(\|\mathbf{x} - \mathbf{x}_j\|). \tag{3.3}$$

We will use the same points as collocation points and data sites for the radial basis function.

For the multiquadric in (2.5) the resulting system of equations is given as

$$\sum_{j=1}^{n} c_j \left\{ \lambda\sqrt{1 + |x_i - x_j|^2} - \int_a^b K(x_i, y)\sqrt{1 + |y - x_j|^2}\, dy \right\} = f(x_i), \qquad i = 1, \ldots, N. \tag{3.4}$$

We approximate the integral in (3.4) with a five point Gauss-Legendre quadrature method. We note that the collocation matrices are not symmetric as is in the case of the RBF interpolation matrices. Moreover, the non-singularity results for interpolation cannot be carried directly to the case of collocation because of the term with the integral. The linear system (3.4) can be written in matrix form as

$$\hat{A}\,\mathbf{c} = \mathbf{f} \tag{3.5}$$

where the matrix $\hat{A}$ with $\mathcal{K}_{ij} \equiv \int_a^b K(x_i, y)\sqrt{1 + |y - x_j|^2}\, dy$ is given as

$$\begin{pmatrix} \lambda\sqrt{1 + |x_1 - x_1|^2} - \mathcal{K}_{11} & \lambda\sqrt{1 + |x_1 - x_2|^2} - \mathcal{K}_{12} & \cdots & \lambda\sqrt{1 + |x_1 - x_N|^2} - \mathcal{K}_{1N} \\ \lambda\sqrt{1 + |x_2 - x_1|^2} - \mathcal{K}_{21} & \lambda\sqrt{1 + |x_2 - x_2|^2} - \mathcal{K}_{22} & \cdots & \lambda\sqrt{1 + |x_2 - x_N|^2} - \mathcal{K}_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda\sqrt{1 + |x_N - x_1|^2} - \mathcal{K}_{N1} & \lambda\sqrt{1 + |x_N - x_2|^2} - \mathcal{K}_{N2} & \cdots & \lambda\sqrt{1 + |x_N - x_N|^2} - \mathcal{K}_{NN} \end{pmatrix}.$$

Moreover,

$$\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix}.$$

4. Numerical Examples and Discussion

In this section, we will solve two numerical examples to show the efficiency of the proposed method. We solve both problems on a uniform mesh on $[a, b]$. To this end we define $h = (b - a)/(N - 1)$ where n is the number of collocation points and so

$$x_i = a + (i - 1)h, \qquad i = 1, \ldots, N.$$

We use $N = 11$ in both examples.

Table 2: Example 1. Errors at the collocation points.

i	Exact Solution	Numerical Solution	Error
1	0.995635	1.000000	$4.3641 \cdot 10^{-3}$
2	0.897046	0.900317	$3.2709 \cdot 10^{-3}$
3	0.800300	0.802411	$2.1129 \cdot 10^{-3}$
4	0.706845	0.707731	$8.8541 \cdot 10^{-4}$
5	0.617822	0.617406	$4.1640 \cdot 10^{-4}$
6	0.534079	0.532281	$1.7979 \cdot 10^{-3}$
7	0.456219	0.452954	$3.2648 \cdot 10^{-3}$
8	0.384633	0.379809	$4.8235 \cdot 10^{-3}$
9	0.319531	0.313050	$6.4805 \cdot 10^{-3}$
10	0.260971	0.252727	$8.2432 \cdot 10^{-3}$
11	0.208885	0.198766	$1.0119 \cdot 10^{-2}$

Table 3: Example 2. Errors at the collocation points.

i	Exact Solution	Numerical Solution	Error
1	0.983006	1.000000	$1.6994 \cdot 10^{-2}$
2	1.090252	1.105171	$1.1491 \cdot 10^{-2}$
3	1.208713	1.221403	$1.2688 \cdot 10^{-2}$
4	1.339568	1.349858	$1.0289 \cdot 10^{-2}$
5	1.484116	1.491825	$7.7083 \cdot 10^{-3}$
6	1.643792	1.648721	$4.9293 \cdot 10^{-3}$
7	1.820182	1.822119	$1.9364 \cdot 10^{-3}$
8	2.015041	2.013753	$1.2884 \cdot 10^{-3}$
9	2.230305	2.225541	$4.7644 \cdot 10^{-3}$
10	2.468116	2.459603	$8.5128 \cdot 10^{-3}$
11	2.730838	2.718281	$1.2556 \cdot 10^{-2}$

4.1. Example 1

Consider the integral equation

$$\lambda u(x) - \int_0^1 e^{xy} u(y)\, dy = f(x), \qquad 0 \le x \le 1, \tag{4.1}$$

with $\lambda = 20$. We use $u(x) = e^{-x}\cos(x)$ and define $f(x)$ accordingly. The results and errors at the collocation points are shown in Table 2 and the plots for the exact and approximate solutions are shown in Figure 1.

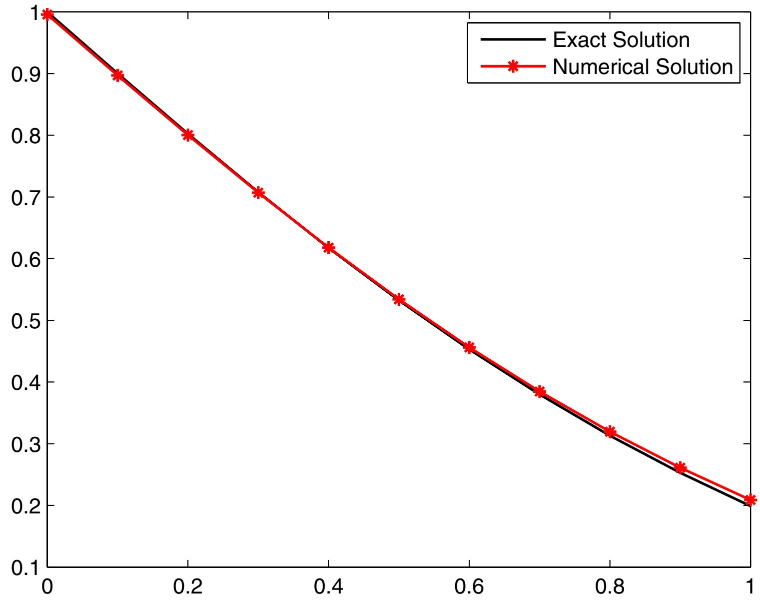

Figure 1: Results for Example 1. Plots of exact solution and numerical solution.

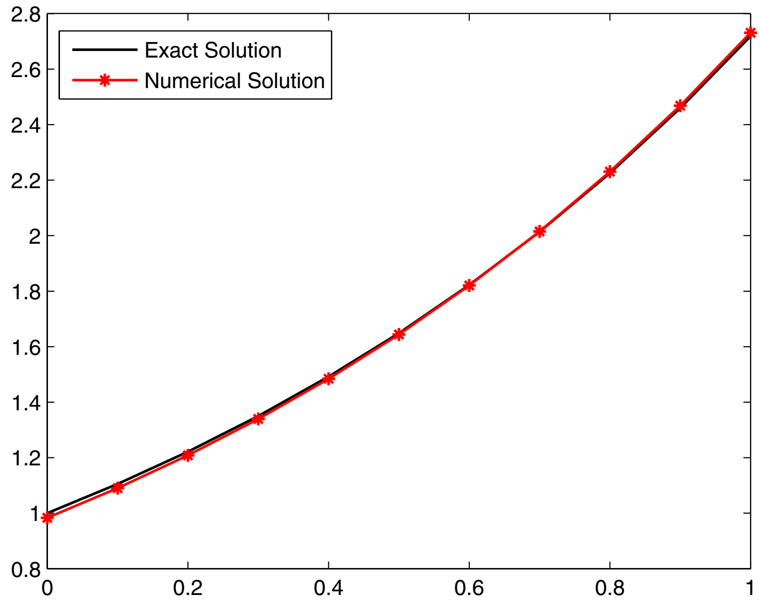

Figure 2: Results for Example 2. Plots of exact solution and numerical solution.

4.2. Example 2

We once again consider (4.1) but with with $\lambda = 50$. Moreover, we use $u(x) = e^x$ and define $f(x)$ accordingly. We show the results and errors at the collocation points in Table 3 and provide the plots of the exact and approximate solutions in Figure 2.

We observe from both examples that the multiquadric method provides good approximation to the solution of the Fredholm integral equation. We however observe an increase in error towards the the boundary of the interval [0, 1] and this may be a direction of further investigation. The conditioning of the collocation matrix will also need to be improved especially as the number of collocation points are increased.

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