

Positive Solutions for Nonlinear Singular Higher Order Boundary Value Problem

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Abstract

In this paper, we investigate the problem of existence of positive solutions for the nonlinear n th order boundary value problem:

$$u^{(n)}(t) + \lambda a(t)f(u(t)), \quad 0 < t < 1, \quad ,$$

$$u(0) = u'(0) = u''(0) = \dots = u^{(n-1)}(0) = 0, \quad \alpha u'(1) + \beta u''(1) = 0.$$

where λ is a positive parameter. By using Krasnoselskii's fixed point theorem of cone, we establish various results on the existence of positive solutions of the boundary value problem.

Under various assumptions on $a(t)$ and $f(u(t))$, we give the intervals of the parameter λ which yields the existence of the positive solutions. An example is also given to illustrate the main results.

Keywords: n th order, boundary-value problem, Krasnoselskii's fixed-point theorem, Green's function, positive solution.

Introduction

One of the most frequently used tools for proving the existence of positive solutions to the integral equations and boundary value problems is Krasnoselskii's theorem on cone expansion and compression and its norm-type version due to Guo [4]. To the best of our knowledge, Wang [7] is the first one who has used this approach. Ever since this pioneering work was achieved, a lot more research was done in this area. Recently [2,5], used Krasnoselskii's fixed-point theorem to prove some existence results to the nonlinear n th order singular boundary value problem:

The purpose of this paper is to establish the existence of positive solutions to nonlinear n th order boundary value problem:

$$u^{(n)}(t) + \lambda a(t)f(u(t)) = 0, \quad 0 < t < 1, \quad (1.1)$$

$$u(0) = u''(0) = u'''(0) = \dots = u^{(n-1)}(0) = 0, \quad \alpha u'(1) + \beta u''(1) = 0. \quad (1.2)$$

Where $\lambda > 0$ is a positive parameter and $a : (0,1) \rightarrow [0,\infty)$ is continuous and $\int_0^1 a(t) dt > 0$, $f : [0,\infty) \rightarrow [0,\infty)$ is continuous and $\alpha, \beta \geq 0$, $\alpha + \beta > 0$. Here, by a positive solution of the boundary value problem we mean a function which is positive on $(0, 1)$ and satisfies differential equation (1.1) and the boundary condition (1.2).

Preliminaries

In this section, we present some notations and lemmas that will be used in the proof of our main results.

Definition 2.1. Let E be a real Banach space. A nonempty closed set $K \subset E$ is called a *cone* of E if it satisfies the following conditions:

- (1) $x \in K$, $\lambda > 0$ implies $\lambda x \in K$;
- (2) $x \in K, -x \in K$ implies $x = 0$.

Definition 2.2. An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets.

All results are based on the following fixed point theorem of cone expansion-compression type due to Krasnoselskii's. See, for example, [4] and [8].

Theorem 2.1. Let E be a Banach space and $K \subset E$ is a cone in E . Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K$ be a completely continuous operator. In addition suppose either:

- (H1) $\|Tu\| \leq \|u\|, \forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, \forall u \in K \cap \partial\Omega_2$ or
- (H2) $\|Tu\| \leq \|u\|, \forall u \in K \cap \partial\Omega_2$ and $\|Tu\| \geq \|u\|, \forall u \in K \cap \partial\Omega_1$

holds. Then T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 1.1. Let $y \in C[0,1]$ then the boundary value problem

$$u^{(n)}(t) + y(t) = 0, \quad 0 < t < 1, \quad (2.1)$$

$$u(0) = u''(0) = u'''(0) = \dots = u^{(n-1)}(0) = 0, \quad \alpha u'(1) + \beta u''(1) = 0. \quad (2.2)$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s) ds,$$

where

$$G(t, s) = \begin{cases} \frac{\beta}{\alpha} t \frac{(1-s)^{n-3}}{(n-3)!} + t \frac{(1-s)^{(n-2)}}{(n-2)!} - \frac{(t-s)^{n-1}}{(n-1)!}, & 0 \leq s \leq t \leq 1 \\ \frac{\beta}{\alpha} t \frac{(1-s)^{(n-3)}}{(n-3)!} + t \frac{(1-s)^{(n-2)}}{(n-2)!}, & 0 \leq t \leq s \leq 1 \end{cases}$$

Proof: Applying the Laplace transform to Eq. (2.1) in the light of Eq. (2.2) we get

$$s^n u(s) - s^{n-1} u(0) - s^{n-2} u'(0) - \dots - u^{(n-1)}(0) = -y(s) \tag{2.3}$$

The Laplace inversion of Eq. (2.3) gives the solution as:

$$u(t) = \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} y(s) ds + \frac{\beta}{\alpha} \int_0^1 \frac{(1-s)^{n-3}}{(n-3)!} y(s) ds - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s) ds. \tag{2.4}$$

The proof is complete.

It is obvious that

$$G(t, s) \geq 0 \quad \text{and} \quad G(1, s) \geq G(t, s), \quad 0 \leq t, s \leq 1. \tag{2.5}$$

Lemma 2.1. $G(t, s) \geq q(t)G(1, s)$ for $0 \leq t, s \leq 1$, where $q(t) = t$.

Proof: If $t \leq s$, then:

$$\frac{G(t, s)}{G(1, s)} = \frac{t \frac{(1-s)^{n-2}}{(n-2)!} + \frac{\beta}{\alpha} t \frac{(1-s)^{n-3}}{(n-3)!}}{\frac{(1-s)^{n-2}}{(n-2)!} + \frac{\beta}{\alpha} \frac{(1-s)^{n-3}}{(n-3)!}} = \frac{t(1-s)^{n-2} + (n-2) \frac{\beta}{\alpha} t(1-s)^{n-3}}{(1-s)^{n-2} + (n-2) \frac{\beta}{\alpha} (1-s)^{n-3}} = t.$$

If $t \geq s$, then

$$\begin{aligned} \frac{G(t, s)}{G(1, s)} &= \frac{t \frac{(1-s)^{n-2}}{(n-2)!} + \frac{t\beta(1-s)^{n-3}}{\alpha(n-3)!} - \frac{(t-s)^{n-1}}{(n-1)!}}{\frac{(1-s)^{n-2}}{(n-2)!} + \frac{\beta(1-s)^{n-3}}{\alpha(n-3)!} - \frac{(1-s)^{n-1}}{(n-1)!}} \\ &= \frac{(n-2)(n-1)\beta t(1-s)^{n-3} + \alpha t(n-1)(1-s)^{n-2} - \alpha(t-s)^{n-1}}{(n-1)(n-2)\beta(1-s)^{n-3} + \alpha(n-1)(1-s)^{n-2} - \alpha(1-s)^{n-1}} \geq t \end{aligned}$$

The proof is complete.

Solution in the cone

In this section, we will apply Krasnoselskii's fixed-point theorem to the eigenvalue problem (1.1), (1.2). We note that $u(t)$ is a solution of (1.1), (1.2) if and only if

$$u(t) = \lambda \int_0^1 G(t, s) a(s) f(u(s)) ds, \quad 0 \leq t \leq 1. \tag{3.1}$$

For our constructions, we shall consider the Banach space $X = C[0,1]$ equipped with standard norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|, u \in X$. Define a cone P by

$$P = \{u \in X \mid u(t) \geq 0, u(t) \geq q(t)\|u\|, t \in [0,1]\}$$

It is easy to see that if $u \in P$, then $\|u\| = u(1)$. Define an integral operator $T : P \rightarrow X$ by

$$Tu(t) = \lambda \int_0^1 G(t,s)a(s)f(u(s))ds, \quad 0 \leq t \leq 1, \quad u \in P. \quad (3.2)$$

Notice from (2.5) that, for $u \in P$, $Tu(t) \geq 0$ on $[0,1]$ and

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)a(s)f(u(s))ds \\ &\geq \lambda q(t) \int_0^1 G(1,s)a(s)f(u(s))ds \\ &\geq \lambda q(t) \max_{0 \leq t \leq 1} \int_0^1 G(t,s)a(s)f(u(s))ds \\ &= q(t) \|Tu\|. \end{aligned}$$

Thus $T(P) \subset P$. In addition, standard arguments show that T is completely continuous.

Following [3,6], we define some important constants:

$$\begin{aligned} A &= \int_0^1 G(1,s)a(s)q(s)ds, & B &= \int_0^1 G(1,s)a(s)ds, \\ F_0 &= \limsup_{u \rightarrow 0^+} \frac{f(u)}{u}, & f_0 &= \liminf_{u \rightarrow 0^+} \frac{f(u)}{u}, \\ F_\infty &= \limsup_{u \rightarrow +\infty} \frac{f(u)}{u}, & f_\infty &= \liminf_{u \rightarrow +\infty} \frac{f(u)}{u} \end{aligned}$$

Here we assume that $\frac{1}{Af_\infty} = 0$ if $f_\infty = \infty$ and $\frac{1}{BF_0} = \infty$ if $F_0 = 0$ and $\frac{1}{Af_0} = 0$ if

$$f_0 = \infty \text{ and } \frac{1}{BF_\infty} = \infty \text{ if } F_\infty = 0$$

Theorem 3. 1. Suppose that $Af_\infty > BF_0$. Then for each $\lambda \in (\frac{1}{Af_\infty}, \frac{1}{BF_0})$ the problem (1.1) and (1.2) has at least one positive solution.

Proof: By the definition of F_0 , we see that there exists an $l_1 > 0$, such that $f(u) \leq (F_0 + \varepsilon)u$ for $0 < u \leq l_1$. If $u \in P$ with $\|u\| = l_1$, we have

$$\begin{aligned} \|Tu\| &= (Tu)(1) \\ &= \lambda \int_0^1 G(1,s) a(s) f(u(s)) ds \\ &\leq \lambda (F_0 + \varepsilon) \|u\| B \end{aligned}$$

Choose $\varepsilon > 0$ sufficiently small such that $(F_0 + \varepsilon)\lambda B \leq 1$. Then we have $\|Tu\| \leq u$.

Thus if we let $\Omega_1 = \{u \in X \mid \|u\| < l_1\}$, then $\|Tu\| \leq u$ for $u \in P \cap \partial\Omega_1$.

Following Sun [3], we choose $c \in (0, \frac{1}{4})$, such that $\lambda \left((f_\infty - \varepsilon) \int_0^c G(1,s) a(s) q(s) ds \right) \geq 1$. There exists $l_3 > 0$, such that $f(u) \geq (f_\infty - \varepsilon)u$ for $u > l_3$. If $u \in P$ with $\|u\| = l_2$, we have

$$\begin{aligned} \|Tu\| &= (Tu)(1) = \lambda \int_0^1 G(1,s) a(s) f(u(s)) ds \\ &\geq \lambda \int_0^1 G(1,s) a(s) (f_\infty - \varepsilon)u(s) ds \\ &\geq \lambda (f_\infty - \varepsilon) \|u\| A \end{aligned}$$

Choose $\varepsilon > 0$ sufficiently small such that $(f_\infty - \varepsilon)\lambda A \leq 1$. Then we have $\|Tu\| \geq \|u\|$.

Let $\Omega_2 = \{u \in X \mid \|u\| < l_2\}$, then $\Omega_1 \subset \bar{\Omega}_2$ and $\|Tu\| \geq u$ for $u \in P \cap \partial\Omega_2$.

Condition (H1) of Krasnoselskii's fixed-point theorem is satisfied. So there exists a fixed point of T in P . This completes the proof.

Theorem 3. 2. Suppose that $A f_0 > B F_\infty$. Then for each $\lambda \in (\frac{1}{A f_0}, \frac{1}{B F_\infty})$ the problem (1.1), (1.2) has at least one positive solution.

Proof: From the definition of f_0 , we see that there exists an $l_1 > 0$, such that $f(u) \geq (f_0 - \varepsilon)u$ for $0 < u \leq l_1$. If $u \in P$ with $\|u\| = l_1$, we have

$$\begin{aligned} \|Tu\| &= (Tu)(1) \\ &= \lambda \int_0^1 G(1,s) a(s) f(u(s)) ds \\ &\geq \lambda (f_0 - \varepsilon) \|u\| A \end{aligned}$$

Choose $\varepsilon > 0$ sufficiently small such that $(f_0 - \varepsilon)\lambda A \geq 1$, then $\|Tu\| \geq \|u\|$ for

$u \in P \cap \partial\Omega_1$.

By the same method, we can see that, if $u \in P$ with $\|u\| = l_2$, then we have $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_2$.

Condition (H2) of Krasnoselskii's fixed-point theorem is satisfied. So there exists a fixed point of T in P . This completes the proof.

Theorem 3.3. Suppose that $\lambda B f(u) < u$ for $u \in (0, \infty)$. Then the problem (1.1), (1.2) has no positive solution.

Proof: Following [1,6], assume to the contrary that u is a positive solution of (1.1), (1.2). Then

$$u(1) = \lambda \int_0^1 G(1,s) a(s) f(u(s)) ds < \frac{1}{B} \int_0^1 G(1,s) a(s) u(s) ds \leq \frac{u(1)}{B} \int_0^1 G(1,s) a(s) ds \leq u(1).$$

This is a contradiction and completes the proof.

Theorem 3.4. Suppose that $\lambda A f(u) > u$ for $u \in (0, \infty)$. Then the problem (1.1), (1.2) has no positive solution.

Proof: Assume to the contrary that u is a positive solution of (1.1), (1.2). Then

$$u(1) = \lambda \int_0^1 G(1,s) a(s) f(u(s)) ds > \frac{1}{A} \int_0^1 G(1,s) a(s) u(s) ds \geq \frac{\|u\|}{A} \int_0^1 G(1,s) a(s) q(s) ds \geq u(1).$$

This is a contradiction and completes the proof.

Example 3.5: Consider the boundary value problem

$$u^{(5)}(t) + \lambda (10s^2 + 2) \frac{7u^2 + u}{u+1} (8 + \sin u) = 0 \quad (3.3)$$

$$u(0) = u'(0) = u''(0) = u^{(4)}(0) = 0, \quad 7u'(1) + 3u''(1) = 0 \quad (3.4)$$

Then $F_0 = f_0 = 8$, $F_\infty = 63$, $f_\infty = 49$, and $8u < f(u) < 63u$. By direct calculations, we obtain that $A = 0.0957341$ and $B = 0.3047619$. Since the general form for a and B are

$$A = \frac{960 + 2n(120 + n(19 + 3n))}{7(3+n)\Gamma[2+n]} \quad \text{if } n > 2$$

$$B = \frac{232 + 2n(31 + n(10 + 3n))}{7(2+n)\Gamma[1+n]} \quad \text{if } n > 2$$

From theorem 3.2 we see that if $\lambda \in (0.2131754, 0.4101562)$ then the problem (3.3), (3.4) has a positive solution. From theorem (3.3) we have that if $\lambda < 0.0520325$ then the problem (3.3), (3.4) has no positive solution. By theorem (3.4), if $\lambda > 0.983415$ then the problem (3.3), (3.4) has no positive solution.

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