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On the ELzaki Transform and Systems of Ordinary Differential Equations

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Abstract

In this paper we introduce some properties and definition of the new integral transform, called ELzaki transform. Farther, we use ELzaki transform to solve systems of ordinary differential equations

Keywords: Elzaki transform-system of differential equations.

Introduction

In the literature there are numerous integral transforms and widely used in physics, astronomy as well as in engineering .The integral transform method is also an efficient method to solve the system of ordinary differential equations. Recently, Tarig ELzaki introduced a new transform and named as ELzaki transform which is defined by the following formula.

$$\mathrm{E}\left[f(t)\right] = v\int_0^\infty f(t) \quad e^{-t/v}dt = T(v) \quad , v \in (-k_1, k_2)$$

And applied this new transform to the solution of system of ordinary differential equations.

In this study, our purpose is to show the applicability of this interesting new transform and its efficiency in solving the linear system of ordinary differential equations.

Tarig ELzaki introduced the ELzaki transform defined by formula

$$T(v) = \mathrm{E}\left[f(t), v\right] = v^2\int_0^\infty f(tv)e^{-t/v}dt \quad , \ k_1 \le v \le k_2 \tag{1}$$

Theorem:

Let $T(v)$ is ELzaki transform of $\left[E\left(f(t)\right)=T(v)\right]$. then:

$$(i)\quad \mathrm{E}\left[f'(t)\right]=\frac{T(v)}{v}-v\,f(0)\quad (ii)\quad \mathrm{E}\left[f''(t)\right]=\frac{T(v)}{v^2}-f(0)-vf'(0)$$

$$(iii)\quad \mathrm{E}\left[f^{(n)}(t)\right]=\frac{T(v)}{v^n}-\sum_{k=0}^{n-1}v^{2-n+k}f^{(k)}(0)$$

Proof:

$(i)\ \mathrm{E}\left[f'(t)\right]=v\int_0^\infty f'(t)\,e^{\frac{-t}{v}}\,dt$ Integrating by parts to find that:

$$\mathrm{E}\left[f'(t)\right]=\frac{T(v)}{v}-v\,f(0)$$

(ii) Let $g(t)=f'(t)$, then: $\mathrm{E}\left[g'(t)\right]=\frac{1}{v}\ \mathrm{E}\left[g(t)\right]-v\,g(0)$

By using (i) we find that:

$$\mathrm{E}\left[f''(t)\right]=\frac{T(v)}{v^2}-f(0)-vf'(0)$$

(iii) Can be proof by mathematical induction.

System of Ordinary Differential Equations

ELzaki transform method is very effective for the solution of the response of a linear system governed by an ordinary differential equation to the initial data and | or to an external disturbance (or external input function). More precisely, we seek the solution of a linear system for its state at subsequent time $t>0$ due to the initial state at $t=0$ and / or to the disturbance applied for $t>0$.

Example I: (System of First Order Ordinary Differential Equations)

Consider the system

$$\begin{cases}\dfrac{dx_1}{dt}=a_{11}x_1+a_{12}x_2+b_1(t)\\[2mm] \dfrac{dx_2}{dt}=a_{21}x_1+a_{22}x_2+b_2(t)\end{cases}\qquad (2)$$

With the initial data. $x_1(0)=x_{10}$ And $x_2(0)=x_{20}$

Where $a_{11},a_{12},a_{21},a_{22}$ are constants

Introducing the matrices

$$x=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ , \ \frac{dx}{dt}=\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} \ , \ A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$b(t)=\begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} and \qquad x_0=\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

We can write the above system in a matrix differential system as

$$\frac{dx}{dt}= Ax+b(t) \quad , \quad x(0)=x_0 \ (3)$$

We take ELzaki transform of the system with the initial conditions to get.

$$\begin{cases} (1-a_{11}v)\bar{x}_1 - a_{12}v\bar{x}_2 = v\bar{b}_1(v)+v^2x_{10} \\ (1-a_{22}v)\bar{x}_2 - a_{21}v\bar{x}_1 = v\bar{b}_2(v)+v^2x_{20} \end{cases}$$

Where $\bar{x}_1$, $\bar{x}_2$, $\bar{b}_1$, $\bar{b}_2$ are Elzaki transform of x_1 , x_2 , b_1 , b_2 respectively.

The solution of these algebraic systems is**,**

$$\bar{x}_1(v)=\frac{\begin{vmatrix} v\bar{b}_1+v^2x_{10} & -a_{12}v \\ v\bar{b}_2+v^2x_{20} & -a_{21}v \end{vmatrix}}{\begin{vmatrix} 1-a_{11}v & -a_{12}v \\ -a_{21}v & 1-a_{22}v \end{vmatrix}} \quad \bar{x}_2(v)=\frac{\begin{vmatrix} 1-a_{11}v & v\bar{b}_1+v^2x_{10} \\ -a_{21}v & v\bar{b}_2+v^2x_{20} \end{vmatrix}}{\begin{vmatrix} 1-a_{11}v & -a_{12}v \\ -a_{21}v & 1-a_{22}v \end{vmatrix}}$$

Expanding these determinants, results for $\bar{x}_1(v)$ and $\bar{x}_2(v)$ can readily be inverted, and the solutions for $x_1(t)$ and $x_2(t)$ can be found in closed forms.

Examples II

Solve the system of differential equations

$$\begin{cases} \frac{dx}{dt}=2x-3y \quad , \quad t>0 \\ \frac{dy}{dt}=y-2x \end{cases}$$

With the initial data

$$x(0)=8 \quad , \quad y(0)=3$$

Take ELzaki transform of the system with the initial conditions to get

$$\begin{cases} \bar{x} - 8v^2 = 2v\bar{x} - 3\bar{y}v \\ \bar{y} - 3v^2 = v\bar{y} - 2\bar{x}v \end{cases}$$

Solve these equations for $\bar{x}$ and $\bar{y}$ we find:

$$\bar{x} = \frac{5\,v^2}{1+v} + \frac{3\,v^2}{1-4\,v}, \qquad \bar{y} = \frac{5\,v^2}{1+v} - \frac{3\,v^2}{1-4\,v}$$

By taking inverse ELzaki transform we have:

$$x = 5\,e^{-t} + 3\,e^{4t}, \qquad y = 5\,e^{-t} - 2\,e^{4t}$$

Examples III

Solve the second order couplet differential system

$$\begin{cases} \dfrac{d^2x}{dt^2} - 3x - 4y = 0 \\ \dfrac{d^2y}{dt^2} + x + y = 0 \end{cases} \qquad t > 0 \quad (4)$$

With initial conditions.

$$x(0) = y(0) = 0 \quad , \quad \frac{dx}{dt}(0) = 2 \quad , \quad \frac{dy}{dt}(0) = 0 \quad (5)$$

The use of the ELzaki transform into (4) with (5) gives

$$\begin{cases} \left(1 - 3\,v^2\right)\bar{x} - 4\,v^2\bar{y} = 2\,v^3 \\ v^2\bar{x} + \left(1 + v^2\right)\bar{y} = 0 \end{cases}$$

Using the same method in example (II) to find the solution in the form:

$$x(t) = t\left[e^t + e^{-t}\right], \qquad y(t) = \frac{1}{2}\left(e^t - e^{-t} - t\,e^t - t\,e^{-t}\right)$$

Examples IV

Consider the following system

$$\begin{cases} y' + z' = t \\ y'' - z = e^{-t} \end{cases}$$

With the initial conditions

$$y(0) = 3 \quad , \quad y'(0) = -2 \quad , \quad z(0) = 0$$

Take ELzaki transform of the system with initial conditions we get

$$\begin{cases} \dfrac{\bar{y}}{v} - 3\ v + \dfrac{\bar{z}}{v} & = v^3 \\ \dfrac{\bar{y}}{v^2} - 3 + 2\ v - \bar{z} & = \dfrac{v^2}{1+v} \end{cases}$$

Where $\bar{z}, \bar{y}$ are ELzaki transform of z, y respectively solve these equations to find $\bar{y}$ and $\bar{z}$.

$$\bar{y} = \frac{v^2}{1+v^2}\left[v^4 + 3\ v^2 - 2\ v + 3 + \frac{v^2}{1+v} \right] \qquad or$$

$$\bar{y} = v^4 + 2\ v^2 + \ \frac{1}{2}\ \frac{v^2}{1+v} - \ \frac{3}{2}\ \frac{v^3}{1+v^2} + \ \frac{1}{2}\ \frac{v^2}{1+v^2}$$

Take the inverse of ELzaki transform, we obtain the solution

$$y = 2 + \frac{1}{2} t^2 + \frac{1}{2} e^{-t} - \frac{3}{2} \sin t + \frac{1}{2} \cos t \ ,$$

and

$$z = y'' - e^{-t} = 1 - \frac{1}{2} e^{-t} + \frac{3}{2} \sin t - \frac{1}{2} \cos t$$

Examples V

Solve the system of second order differential equations.

$$\begin{cases} \dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} - \dfrac{dy}{dt} = 0 \\ 2\dfrac{d^2x}{dt^2} + \dfrac{d^2y}{dt^2} = 6 + 2e^{-t} \end{cases} \tag{6}$$

With the initial conditions

$$x(0) = 1 \quad , \quad x'(0) = -1 \quad , \quad y(0) = 0 \quad , \quad y'(0) = 2$$

Applied ELzaki transform to this equation and making use of the initial conditions we obtain.

$$\frac{\bar{x}}{v^2} - 1 + v + \frac{\bar{x}}{v} - v - \frac{\bar{y}}{v} = 0$$

$$2\frac{\bar{x}}{v^2} - 2 + 2v + \frac{\bar{y}}{v^2} - 2v = 6v^2 + \frac{2v^2}{1+v}$$

Or:

$$\left[\frac{1}{v^2}+\frac{1}{v}\right]\bar{x}-\frac{\bar{y}}{v}=1$$

$$\frac{2\bar{x}}{v^2}+\frac{\bar{y}}{v^2}=6v^2+2+\frac{2v^2}{1+v^2}$$

The solutions of these algebraic equations are

$$\bar{x}=\frac{v^2}{1+3v}\left[6v^2+2v+1+\frac{2v^3}{1+v}\right]$$

We con write this equation in the form,

$$\bar{x}=2v^4+\frac{v^2}{1+v}.$$

Inverting this result, we obtain,

$$x=t^2+e^{-t}.$$

From (6) we can find.

$$\frac{dy}{dt}=\frac{d^2x}{dt^2}+\frac{dx}{dt}=2+2t \quad and \quad y=2t+t^2$$

Conclusion

ELzaki transform provides powerful method for analyzing derivatives. It is heavily used to solve ordinary differential equations, Partial differential equations and system of ordinary differential equations.

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