Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 7, Number 1 (2011), pp. 83–88

http://www.ripublication.com/gjpam.htm

On Infinite Series of Hypergeometric Function of Three Variables

M. Kamarujjama and Waseem A. Khan

Department of Applied Mathematics,
Z.H.College of Engineering and Technology,
Aligarh Muslim University, Aligarh-202002, India.
E-mail: mdkamarujjama@rediffmail.com,
waseem08_khan@rediffmail.com

Abstract

This paper deals with an integral transformation involving Whittaker function $M_{k,m}$ into Lauricella's function F_2. Particular cases of this transformation correspond to a general triple hypergeometric function $F^{(3)}$ and several known and new transformations are also discussed.

AMS Subject Classification: 33C15, 33C20, 33C65.
Keywords: Whittaker function, Horn function, Appell's function and Laplace transform.

1. Introduction and Definition

A Whittaker function $M_{k,m}$ was introduced by Whittaker [5] (see also Whittaker and Watson [6]) in terms of Confluent hypergeometric function ${}_1F_1$ (or Kummer's function)

$$M_{k,m}(x) = x^{m+1/2}e^{(-1/2)x}\,{}_1F_1\left(\frac{1}{2}+m-k, 2m+1; x\right) \tag{1.1}$$

and the Horn's function H_4 [6;p.39] is defined as

$$H_4[\alpha,\beta,\gamma,\delta;x,y] = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_n x^m y^n}{(\gamma)_m(\delta)_n m!n!} \tag{1.2}$$

where $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$, $|\,x\,|< r$, $|\,y\,|< s$ and $4r = (s-1)^2$.

A general triple hypergeometric series $F^{(3)}[x, y, z]$ of Srivastava [7; p.69(39)] defined as

$$F^{(3)}\left[\begin{array}{lll}(a) & :: (b); (b'); (b''); (c); (c'); (c'') & ; \\ & & x, y, z \\ (e) & :: (g); (g'); (g''); (h); (h'); (h'') & ;\end{array}\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{((a))_{m+n+p}((b))_{m+n}((b'))_{n+p}((b''))_{p+m}((c))_m((c'))_n((c''))_p}{((e))_{m+n+p}((g))_{m+n}((g'))_{n+p}((g''))_{p+m}((h))_m((h'))_n((h''))_p} \frac{x^m\ y^n\ z^p}{m!\ n!\ p!}, \tag{1.3}$$

with, as usual, (α) abraviates the array of A-parameters

$$\alpha_1, \alpha_2 \cdots \alpha_A, \quad ((\alpha))_m = \prod_{j=1}^{A} (\alpha_j)_m = \prod_{j=1}^{A} \frac{\Gamma(\alpha_j + m)}{\Gamma(\alpha_j)}. \tag{1.4}$$

2. Integral Transform

We establish the following integral

$$\int_0^\infty t^{\sigma-1/2} e^{-pt}\, {}_1F_2\left(\begin{array}{ll}\eta & ; \\ & -x^2t^2 \\ \gamma, \delta & ;\end{array}\right) M_{\kappa_1,\mu_1-1/2}(\beta_1 t) M_{\kappa_2,\mu_2-1/2}(\beta_2 t)dt$$

$$= \sum_{r=0}^{\infty} \frac{(\eta)_r(-1)^r x^{2r} \beta_1^{\mu_1} \beta_2^{\mu_2} \Gamma(a+2r)}{(\gamma)_r(\delta)_r r!(p + \frac{\beta_1}{2} + \frac{\beta_2}{2})^{a+2r}}$$

$$F_2\left[a+2r, \mu_1 - k_1, \mu_2 - k_2, 2\mu_1, 2\mu_2; \frac{\beta_1}{p + \frac{\beta_1}{2} + \frac{\beta_2}{2}}, \frac{\beta_2}{p + \frac{\beta_1}{2} + \frac{\beta_2}{2}}\right] \tag{2.1}$$

where $a = \sigma + \mu_1 + \mu_2 + \frac{1}{2}$, $Re(p + \frac{\beta_1}{2} + \frac{\beta_2}{2}) > 0$ and $Re(a) > 0$ and Appell's function F_2 [7;p.53] is defined as

$$F_2\left[a, b, b'; c, c'; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_m(c')_n m!n!}.$$

Equation (2.1) can be established by expanding ${}_1F_2$ in series and integrating term by term with the help of a result [3;p.216(14)].

Setting $k_1 = k_2 = 0$ in (2.1), using a result of Bailey [10;p.11] involving F_2 and F_4 [2;p.224]

$$F_2\left[\alpha, \beta - \frac{1}{2}, \beta' - \frac{1}{2}, 2\beta - 1, 2\beta' - 1; 2x, 2y\right]$$

$$= (1-x-y)^{-\alpha} F_4\left[\frac{1}{2}\alpha, \frac{1}{2}\alpha+\frac{1}{2}, \beta, \beta'; \frac{x^2}{(1-x-y)^2}, \frac{y^2}{(1-x-y)^2}\right] \tag{2.2}$$

Expanding F_4 in series and making an appeal of Legendre's duplication formula [2;p.5(15)]

$$\left(\frac{1}{2}a\right)_m \left(\frac{1}{2}a+\frac{1}{2}\right)_m = 2^{-2m}(a)_{2m}, m = 0, 1, 2, \cdots$$

in conjugation with (1.3), we get

$$\int_0^\infty t^{\sigma-1/2} e^{-pt} {}_1F_2\left(\begin{matrix} \eta & ; & \\ & & -x^2t^2 \\ \gamma, \delta & ; & \end{matrix}\right) M_{0,\mu_1-1/2}(\beta_1 t) M_{0,\mu_2-1/2}(\beta_2 t)\,dt$$

$$= \frac{\beta_1^{\mu_1}\beta_2^{\mu_2}\Gamma(a)}{p^a} F^{(3)}\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} & :: \_\_\_\_ : \eta; \_; \_\_\_; \_\_\_; & \\ & & \frac{-4x^2}{p^2}, \frac{\beta_1^2}{4p^2}, \frac{\beta_2^2}{4p^2} \\ \_\_\_\_ & :: \_\_\_\_ : \gamma, \delta; \mu_1+\frac{1}{2}; \mu_2+\frac{1}{2}; & \end{matrix}\right] \tag{2.3}$$

where $a = \sigma+\mu_1+\mu_2+\frac{1}{2}$, $Re(a) > 0$ and $Re\left(p+\frac{\beta_1}{2}+\frac{\beta_2}{2}\right) > 0$.

Comparing of (2.3) to (2.1) with $k_1 = k_2 = 0$ would yield an expansion

$$\sum_{r=0}^{\infty} \frac{(\eta)_r(-1)^r(\alpha)_{2r}}{(\gamma)_r(\delta)_r r!} F_2\left[\alpha+2r, \sigma, \omega, 2\sigma, 2\omega; y, z\right]$$

$$= \left(1-\frac{1}{2}y-\frac{1}{2}z\right)^{-\alpha} F^{(3)}\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} & :: \_\_\_ : \eta; \_; \_\_\_; \_\_\_; & \\ \_\_\_\_ & :: \_\_\_ : \gamma, \delta; \sigma+\frac{1}{2}; \omega+\frac{1}{2}; & \frac{-16x^2}{(2-y-z)^2}, \frac{y^2}{(2-y-z)^2}, \\ & & \frac{z^2}{(2-y-z)^2} \end{matrix}\right] \tag{2.4}$$

In view of a known transformation [1;p.381].

$$F_2\left[\alpha, \beta, \beta', 2\beta, 2\beta'; 2x, y\right] = (1-x)^{-\alpha} H_4\left[\alpha, \beta', \beta+\frac{1}{2}, 2\beta'; \frac{x^2}{4(1-x)^2}, \frac{y}{1-x}\right], \tag{2.5}$$

equation (2.4) gives an infinite sum of H_4 in the form

$$\sum_{r=0}^{\infty} \frac{(\eta)_r(-1)^r(\alpha)_{2r}}{(\gamma)_r(\delta)_r r!} \left(1-\frac{y}{2}\right)^{-2r} H_4\left[\alpha+2r, \omega, \sigma+\frac{1}{2}, 2\omega; \frac{y^2}{4(2-y)^2}, \frac{2z}{2-y}\right]$$

$$= \left(\frac{2-y-z}{2-y}\right)^{-\alpha} F^{(3)} \left[\begin{array}{ll} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} & :: \_\_\_ : \eta; \_; \_\_\_; \_\_\_; \\ & \qquad \frac{-16x^2}{(2-y-z)^2}, \frac{y^2}{(2-y-z)^2}, \\ \_\_\_\_\_\_ & :: \_\_\_ : \gamma, \delta; \sigma+\frac{1}{2}; \omega+\frac{1}{2}; \\ & \qquad \frac{z^2}{(2-y-z)^2} \end{array}\right] \tag{2.6}$$

For $z \to 0$, equation (2.4) gives

$$\sum_{r=0}^{\infty} \frac{(\eta)_r(-1)^r(\alpha)_{2r}}{(\gamma)_r(\delta)_r r!} {}_2F_1\left(\begin{array}{ll} \alpha+2r, \sigma & ; \\ & \quad y \\ 2\omega & ; \end{array}\right) x^{2r}$$

$$= \left(1-\frac{1}{2}y\right)^{-\alpha} F^{2:1;0}_{0:2;1} \left[\begin{array}{ll} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} & :: \_\_\_\_ : \eta; \_; \_\_\_; \_\_\_; \\ & \qquad \frac{-16x^2}{(2-y)^2}, \frac{y^2}{(2-y)^2} \\ \_\_\_\_\_ & :: \_\_\_\_ : \gamma, \delta; \sigma+\frac{1}{2}; \_\_\_\_; \end{array}\right] \tag{2.7}$$

$$\sum_{r,k=0}^{\infty} \frac{(\eta)_r(-1)^r(\alpha)_{2r+k}(\sigma)_k y^k x^{2r}}{(\gamma)_r(\delta)_r(2\omega)_k r!k!}$$

$$= \left(1-\frac{1}{2}y\right)^{-\alpha} F^{2:1;0}_{0:2;1} \left[\begin{array}{ll} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} & :: \_\_\_\_ : \eta; \_; \_\_\_; \_\_\_; \\ & \qquad \frac{-16x^2}{(2-y)^2}, \frac{y^2}{(2-y)^2} \\ \_\_\_\_ & :: \_\_\_\_ : \gamma, \delta; \sigma+\frac{1}{2}; \_\_\_\_; \end{array}\right] \tag{2.8}$$

On setting $\eta = \gamma$, the result (2.8) reduces to known result Pathan [9;p.41].

On setting $k_2 = -\mu_2$ in (2.1), the result reduces to

$$\beta_2^{\mu_2} \int_0^{\infty} t^{\sigma+\mu_2-1/2} e^{-(p-1/2\beta_2)t} {}_1F_2\left(\begin{array}{ll} \eta & ; \\ & \quad -x^2t^2 \\ \gamma, \delta & ; \end{array}\right) M_{\kappa_1,\mu_1-1/2}(\beta_1 t)dt$$

$$= \beta_1^{\mu_1}\beta_2^{\mu_2} \sum_{r=0}^{\infty} \frac{(\eta)_r(-1)^r x^{2r}\Gamma(a+2r)}{(\gamma)_r(\delta)_r r!(p+\frac{\beta_1}{2}-\frac{\beta_2}{2})^{a+2r}} {}_2F_1\left[a+2r, \mu_1-k_1, 2\mu_1; \frac{\beta_1}{p+\frac{\beta_1}{2}-\frac{\beta_2}{2}}\right]$$

$$= \frac{\beta_1^{\mu_1}\beta_2^{\mu_2}\Gamma(a)}{(p+\frac{\beta_1}{2}-\frac{\beta_2}{2})^a}\sum_{r,s=0}^{\infty}\frac{(\eta)_r(-1)^r(a)_{2r+s}(\mu_1-k_1)_s x^{2r}y^s}{(\gamma)_r(\delta)_r(p+\frac{\beta_1}{2}-\frac{\beta_2}{2})^{2r}r!s!} \tag{2.9}$$

On setting $\eta = \gamma, \delta = \mu, x \to \frac{x}{2}$ and $x = 2\alpha$ in (2.9), the result reduces to known result Pathan [9;p.42].

3. Series Expansion

The generalized hypergeometric series of a power of t is given by [4;p.417].

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}{}_{p+1}F_q\left(\begin{matrix}-k,(a_p) & ; & \\ & & x\\ (b_q) & ; & \end{matrix}\right) = e^t{}_pF_q\left(\begin{matrix}(a_p) & ; & \\ & & -tx\\ (b_q) & ; & \end{matrix}\right) \tag{3.1}$$

For $p = 1, q = 2$, equation (3.1) reduces to

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}{}_2F_2\left(\begin{matrix}-k,(a_1) & ; & \\ & & x\\ b_1,b_2 & ; & \end{matrix}\right) = e^t{}_1F_2\left(\begin{matrix}a_1 & ; & \\ & & -tx\\ b_1,b_2 & ; & \end{matrix}\right) \tag{3.2}$$

Multiplying both the sides of (3.2) $t^{\sigma-1/2}e^{-pt}M_{0,\mu_1-1/2}(\beta_1 t)M_{0,\mu_2-1/2}(\beta_2 t)$ and integrating with respect to t from zero to infinity with the help of the result (2.3) and [3;p.216(14)], we obtain

$$\sum_{k=0}^{\infty}\frac{(a)_k}{p^k k!}F^{(3)}\left[\begin{matrix}\_ :: \frac{1}{2}(a+k),\frac{1}{2}(a+k)+\frac{1}{2};\_ : -k,a_1;\_\_\_;\_\_\_; & \\ & x,\frac{\beta_1^2}{4p^2},\frac{\beta_2^2}{4p^2}\\ \_ :: \_\_\_\_\_\_\_\_ : b_1,b_2;\mu_1+\frac{1}{2};\mu_2+\frac{1}{2}; & \end{matrix}\right]$$

$$= \left(\frac{p}{p-1}\right)^a\sum_{m=0}^{\infty}\frac{(-1)^m(a_1)_m(a)_m x^m}{(b_1)_m(b_2)_m(p-1)^m m!}F^{(2)}$$

$$\times\left[\begin{matrix}\frac{1}{2}(a+m),\frac{1}{2}(a+m)+\frac{1}{2}:\_\_\_;\_\_\_; & \frac{\beta_1^2}{4(p-1)^2},\frac{\beta_2^2}{4(p-1)^2}\\ & \\ \_\_\_\_\_\_\_\_ : \mu_1+\frac{1}{2};\mu_2+\frac{1}{2}; & \end{matrix}\right] \tag{3.3}$$

On setting $x = 0$ in (3.3), the result reduces to

$$\sum_{k=0}^{\infty}\frac{(a)_k}{p^k k!}F^{(2)}\left[\begin{matrix}\_ :: \frac{1}{2}(a+k),\frac{1}{2}(a+k)+\frac{1}{2};- : -;-;\_\_\_;\_\_\_; & \\ & \frac{\beta_1^2}{4p^2}\frac{\beta_2^2}{4p^2}\\ \_ :: \_\_\_\_\_\_\_\_ : -;-;\mu_1+\frac{1}{2};\mu_2+\frac{1}{2}; & \end{matrix}\right]$$

$$= \left(\frac{p}{p-1}\right)^{a} F^{(2)} \left[\begin{array}{r} \frac{1}{2}(a+m), \frac{1}{2}(a+m)+\frac{1}{2} : \_\_\_; \_\_\_; \qquad\qquad\qquad \\ \frac{\beta_1^2}{4(p-1)^2}, \frac{\beta_2^2}{4(p-1)^2} \\ \_\_\_\_\_\_\_\_\_\_\_\_ : \mu_1+\frac{1}{2}; \mu_2+\frac{1}{2}; \qquad\qquad\qquad \end{array}\right] \tag{3.4}$$

References

[1] A. Erdelyi, et al., Transformation of hypergeometric functions of two variables, Proc. Roy. Soc. Edinburgh Sect. A, 62 (1948), 378–385.

[2] A. Erdelyi, et al., Higher transcendental functions, Vol.I, McGraw-Hill New york, 1953.

[3] A. Erdelyi, et al., Table of Integral Transfoms,Vol.I, McGraw-Hill New york, 1954.

[4] A.P.Prudnikov, et al., Integrals and Series, Vol.3, Gorden and Breach Science Publishers, New York, 1990.

[5] E.T. Whittaker., An expression of certain known function as generalized hypergeometric functions, Bull. Amer. Math. Soc., 10 (1903), 125–134.

[6] E.T. Whittaker and G.N.Watson., ACourse in Modern Analysis, 4th ed. Cambridge, England, Cambridge University Press, 1990.

[7] H.M. Srivastava and H.L. Manocha., A treatise on generating functions, Halsted Press (Ellis Horwood Ltd., Chichester), JohnWiley and Sons, New York, 1984.

[8] M.A. Pathan., Finite summation of Horn's function H_4, Bull. Inst. Math. Acad. Sinica, 6 (1978), 67–71.

[9] M.A. Pathan., On infinite series of Horn's function H_4, Bull. Inst. Math. Acad. Sinica, 8 (1980), 39–44.

[10] W.N. Bailey., The generating function of Jacobi polynomials, J. London Math. Soc., 13 (1938), 8–12.