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Unbranched Riemann domains over Stein spaces

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Abstract

In this article, we show that if $\Pi : X \to \Omega$ is an umbranched Riemann domain with Ω Stein and Π a locally 1-complete morphism, then X is Stein. This gives in particular a positive answer to the local Steiness problem, namely if X is a Stein space and, if Ω is a locally Stein open set in X, then Ω is Stein.

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1. Introduction

A holomorphic map $\Pi : X \to Y$ of complex spaces is said to be a locally r-complete morphism if for every $x \in Y$, there exists an open neighborhood U of x such that $\Pi^{-1}(U)$ is r-complete. When $r = 1$, Π is called locally 1-complete or locally Stein morphism.

In [3] Coltoiu and Diederich proved the following

Theorem 1.1. Let X and Y be complex spaces with isolated singularities and $\Pi : X \to Y$ an umbranched Riemann domain such that Y is Stein and Π is a locally Stein morphism. Then X is Stein.

In this article, we prove that the same result follows if we assume only that Y is an arbitrary Stein space.

An immediate consequence of this result is the

Corollary 1.2. Let X be a Stein space, and let $\Omega \subset X$ be an open subset which is locally Stein in the sense that every point $x \in \partial\Omega$ has an open neighborhood U in X such that $U \cap \Omega$ is Stein. Then Ω is itself Stein.

2. Preliminaries

We start by recalling some definitions which are important for our purposes.

Let Ω be an open set in $\mathbb{C}^n$ with complex coordinates $z_1, \cdots, z_n$. Then it is known that a function $\phi \in C^{\infty}(\Omega)$ is q-convex if for every point $z \in \Omega$, there exists a complex vector subspace $E(z)$ of $\mathbb{C}^n$ of dimension at least $n - q + 1$ such that the levi form $L_z(\phi, \xi) = \sum_{i,j} \frac{\partial^2 \phi(z)}{\partial z_i \partial \overline{z}_j} \xi_i \overline{\xi}_j$ is positive definite at each point $\xi \in E(z)$.

A smooth real valued function ϕ on a complex space X is called q-convex if every point $x \in X$ has an open neighborhood U isomorphic to a closed analytic set in a domain $D \subset \mathbb{C}^n$ such that the restriction $\phi|_U$ has an extension $\tilde{\phi} \in C^{\infty}(D)$ which is q-convex on D.

We say that X is q-complete if there exists a q-convex function $\phi \in C^{\infty}(X, \mathbb{R})$ which is exhaustive on X i.e. $\{x \in X : \phi(x) < c\}$ is relatively compact for any $c \in \mathbb{R}$.

The space X is said to be cohomologically q-complete if for every coherent analytic sheaf $\mathcal{F}$ on X the cohomology groups $H^p(X, \mathcal{F})$ vanish for all $p \geq q$.

An open subset D of Ω is called q-Runge if for every compact set $K \subset D$, there is a q-convex exhaustion function $\phi \in C^{\infty}(\Omega)$ such that

$$K \subset \{x \in \Omega : \phi(x) < 0\} \subset\subset D$$

It is shown in [2] that if D is q-Runge in Ω, then for every $\mathcal{F} \in coh(\Omega)$ the cohomology groups $H^p(D, \mathcal{F})$ vanish for $p \geq q$ and, the restriction map

$$H^p(\Omega, \mathcal{F}) \longrightarrow H^p(D, \mathcal{F})$$

has dense image for all $p \geq q - 1$.

3. Main result

Lemma 3.1. Let X and Y be complex spaces and $\Pi : X \to Y$ an unbranched Riemann domain. Assume that there exists a smooth q-convex function ϕ on Y. Then, for any real number c and every coherent analytic sheaf $\mathcal{F}$ on X if $dih(\mathcal{F}) > q$ and $X'_c = \{x \in X : \phi o \Pi(x) > c\}$, the restriction map $H^p(X, \mathcal{F}) \to H^p(X'_c, \mathcal{F})$ is an isomorphism for $p \leq dih(\mathcal{F}) - q - 1$.

Here $dih(\mathcal{F})$ denotes the homological dimension of $\mathcal{F}$.

Let V be a closed analytic set in a domain $D \subset \mathbb{C}^n$ and $\phi \in C^{\infty}(V)$ a q-convex function. Let $\xi \in V$ and suppose we can find a q-convex function $\hat{\phi} \in C^{\infty}(D)$ with $\hat{\phi}|_V = \phi$ and that n is equal to the dimension of the Zariski tangent space at ξ.

Then in order to prove lemma 1 we shall need the following result due to Andreotti-Grauert [2].

Theorem 3.2. For any coherent analytic sheaf $\mathcal{F}$ on V with $dih(\mathcal{F}) > q$, there exists a fundamental system of Stein neighborhoods $U \subset D$ of ξ such that if $Y = \{z \in V :$

$\phi(z) > 0\}$, then $H^p(Y \cap U, \mathcal{F}) = 0$ for $0 < p < dih_\xi(\mathcal{F}) - q$ and $H^0(U \cap V, \mathcal{F}) \to H^p(U \cap Y, \mathcal{F})$ is an isomorphism.

Proof. Let $\xi \in X$ such that $\phi o\Pi(\xi) = c$, and let $V \subset\subset X$ be a hyperconvex open neighborhood of ξ, biholomorphic by Π to the open subset $W = \Pi(V) \subset Y$. We may take V so that W is biholomorphic to a closed analytic subset of a domain D in $\mathbb{C}^n$ of minimal dimension and $\phi|_W$ extends to a smooth q-convex function in a neighborhood $W_1 \subset D$ of W. Let $\psi : V \to]-\infty, 0[$ be a continuous strictly plurisubharmonic function. Then it is clear that $\psi_k = \frac{1}{k}\psi + \phi o\Pi$, $k \geq 1$, is an increasing sequence of q-convex functions on V. If we put $V_k = \{x \in V : \psi_k(x) > c\}$, then $\bigcup_{k\geq 1} V_k = V \cap X'_c$. Moreover, for any $x \in V$, there exists, by theorem 2, a fundamental system of connected Stein neighborhoods $U \subset V$ such that $H^r(U \cap V_k, \mathcal{F}) = 0$ for $1 \leq r < dih(\mathcal{F}) - q$ and $H^o(U, \mathcal{F}) \to H^o(U \cap V_k, \mathcal{F})$ is an isomorphism, or equivalently (See [4] or [1]) $\underline{H^r_{S_k}}(\mathcal{F}) = 0$ for $r \leq dih(\mathcal{F}) - q$, where $\underline{H^r_{S_k}}(\mathcal{F})$ is the cohomology sheaf with support in $S_k = \{x \in V : \psi_k(x) \leq c\}$ and coefficients in $\mathcal{F}$. Furthermore, there exists a spectral sequence

$$H^p_{S_k}(V, \mathcal{F}) \Longleftarrow E_2^{p,q} = H^p(V, \underline{H^p_{S_k}}(\mathcal{F}))$$

Since $\underline{H^p_{S_k}}(\mathcal{F}) = 0$ for $p \leq dih(\mathcal{F}) - q$, then for any $p \leq dih(\mathcal{F}) - q$ the cohomology groups $H^p_{S_k}(V, \mathcal{F}) =$ vanish and, the exact sequence of local cohomology

$$\cdots \to H^p_{S_k}(V, \mathcal{F}) \to H^p(V, \mathcal{F}) \to H^p(V_k, \mathcal{F}) \to H^{p+1}_{S_k}(V, \mathcal{F}) \to \cdots$$

implies that $H^p(V_k, \mathcal{F}) \cong H^p(V, \mathcal{F})$ for all $p \leq dih(\mathcal{F}) - q - 1$. Hence $H^p(V_k, \mathcal{F}) = 0$ for $1 \leq p \leq dih(\mathcal{F}) - q - 1$ and $H^o(V_k, \mathcal{F}) \cong H^o(V, \mathcal{F})$ for every integer k. Since $V \cap X'_c$ is an increasing union of V_k, $k \in \mathbb{N}$, then, by ([2], lemma, p. 250), we deduce that $H^p(V \cap X'_c, \mathcal{F}) = 0$ for $1 \leq p \leq n - q - 1$ and $H^o(V, \mathcal{F}) \to H^o(V \cap X'_c, \mathcal{F})$ is an isomorphism. Since each point of X has a fundamental system of hyperconvex neighborhoods, then, if $S = \{x \in X : \phi o\Pi(x) \leq c\}$, the cohomology sheaf $H^p_S(\mathcal{F})$ vanishes for all $p \leq dih(\mathcal{F}) - q$. Therefore the spectral sequence

$$H^p_S(X, \mathcal{F}) \Longleftarrow E_2^{p,q} = H^p(X, \underline{H^p_S}(\mathcal{F}))$$

shows that $H^p_S(X, \mathcal{F}) = 0$ for any $p \leq dih(\mathcal{F}) - q$, and from the exact sequence

$$\cdots \to H^p_S(X, \mathcal{F}) \to H^p(X, \mathcal{F}) \to H^p(X'_c, \mathcal{F}) \to H^{p+1}_S(X, \mathcal{F}) \cdots$$

we see that $H^r(X, \mathcal{F}) \cong H^r(X'_c, \mathcal{F})$ for any $c \in \mathbb{R}$ and all $r \leq n - q - 1$. ■

Lemma 3.3. Let Y be a Stein space and $\Pi : X \to Y$ an unbranched Riemann domain and locally r-complete morphism. Then for any coherent analytic sheaf $\mathcal{F}$ on X with $dih(\mathcal{F}) \geq r + 2$, the cohomology group $H^p(X, \mathcal{F}) = 0$ for all $p \geq r$.

Proof. Since $\Pi : X \to Y$ is locally r-complete, it follows from [7] that $H^p(X, \mathcal{F}) = 0$ for all $p \geq r + 1$. It is therefore enough to prove that $H^r(X, \mathcal{F}) = 0$.

We consider a covering $\mathcal{V} = (V_i)_{i\in\mathbb{N}}$ of Y by open sets $V_i \subset \Omega$ such that $\Pi^{-1}(V_i)$ is r-complete for all $i \in \mathbb{N}$. By the Stein covering lemma of Sthelé [6], there exists a locally finite covering $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ of Y by Stein open subsets $U_i \subset\subset Y$ such that $\mathcal{U}$) is a refinement of $\mathcal{V}$, $\bigcup_{i\leq j} U_i$ is Stein for all is Stein for all j. Moreover, there exists for all $j \in \mathbb{N}$ a continuous strictly plurisubharmonic function ϕ_{j+1} on $\bigcup_{i\leq j+1} U_i$ such that

$$\bigcup_{i\leq j} U_i \cap U_{j+1} = \{x \in U_{j+1} : \phi_{j+1}(x) < 0\}$$

Note also that $\Pi^{-1}(U_i)$ is r-complete for all $i \in \mathbb{N}$ and, if $X_j = \Pi^{-1} \bigcup_{i\leq j} U_i$ and $X'_{j+1} = \Pi^{-1}(U_{j+1})$, then $X_j \cap X'_{j+1} = \{x \in X'_{j+1} : \phi_{j+1} o\Pi(x) < 0\}$ is clearly r-Runge in X'_{j+1}.

We shall first prove by induction on j that $H^r(X_j, \mathcal{F}) = 0$. For $j = 0$, this is clear, since $\Pi^{-1}(U_o)$ is r-complete. Assume that $j \geq 1$, $H^r(X_j, \mathcal{F}) = 0$ and put $Y_j = \{x \in X_j : \phi_{j+1} o\Pi(x) > 0\}$ and $Y'_{j+1} = \{x \in X'_{j+1} : \phi_{j+1} o\Pi(x) > 0\}$. Then, by lemma 1, $H^p(Y_j, \mathcal{F}) \cong H^p(X_j, \mathcal{F})$ and $H^p(Y'_{j+1}, \mathcal{F}) \cong H^p(X'_{j+1}, \mathcal{F})$ for $p \leq r$. Since $Y_{j+1} = \{x \in X_{j+1} : \phi_{j+1} o\Pi(x) > 0\} = Y_j \cup Y'_{j+1}$ and $Y_j \cap Y'_{j+1} = \emptyset$, then we have

$$H^p(X_{j+1}, \mathcal{F}) \cong H^p(Y_{j+1}, \mathcal{F}) \cong H^p(Y_j, \mathcal{F}) \oplus H^p(Y'_{j+1}, \mathcal{F}) \ \ for \ \ all \ \ p \leq r$$

This proves in particular that $H^r(X_j, \mathcal{F}) = 0$ for all $j \in \mathbb{N}$.

Moreover, since X is an increasing union of $(X_j)_{j\geq 0}$ and $H^{r-1}(X_{j+1}, \mathcal{F}) \cong H^{r-1}(X_j, \mathcal{F}) \oplus H^{r-1}(X'_{j+1}, \mathcal{F})$, then, by [2, lemma, p. 250], the restriction map $H^r(X, \mathcal{F}) \to H^r(X_0, \mathcal{F})$ is an isomorphism, which implies that $H^r(X, \mathcal{F}) = 0$. ■

Theorem 3.4. Let $\Pi : X \to Y$ be an unbranched Riemann domain with Y a Stein space of dimension n and Π a locally Stein morphism. Then X is Stein.

Proof. The proof is by induction on the dimension of Y.

In order to prove theorem 3 we have only to verify that $H^1(X, \mathcal{O}_X) = 0$. (See [5]). Suppose that $n = 2$, and let $\xi : \tilde{Y} \to Y$ be a normalization of Y. If $\tilde{X}$ denotes the fiber product of $\Pi : X \to Y$ and the normalization $\xi : \tilde{Y} \to Y$, then $\tilde{X} = \{(x, \tilde{y}) \in X \times \tilde{Y} : \Pi(x) = \xi(\tilde{y})\}$ and, it is clear that the projection $\Pi_2 : \tilde{X} \to \tilde{Y}$ is an unbranched Riemann domain over the 2-dimensional Stein normal space $\tilde{Y}$. Moreover, since Π_2 is obviously a locally Stein morphism, it follows from [3] that $\tilde{X}$ is Stein. On the other hand, it is easy to verify that the projection $\Pi_1 : \tilde{X} \to X$ is a finite holomorphic surjection, which implies that X is Stein. (See e.g. [8]).

We now suppose that $n \geq 3$ and that the theorem has already proved if $dim(Y) \leq n-1$.

Since a complex space X is Stein if and only if each irreducible component X_i of X is Stein, then we may assume that Y is irreducible.

Let f be a holomorphic function on Y, $f \neq 0$, but $Z = \{f = 0\} \neq \emptyset$. Then $\Pi|_{Z'} : Z' = \Pi^{-1}(Z) \to Z$ is an unbranched Riemann domain and locally Stein. Therefore Z' is Stein by the induction hypothesis. Furthermore, if $\mathcal{I}(Z')$ denotes the ideal sheaf of Z', it follows from [2] that $dih(\mathcal{I}(Z') = dih(\mathcal{O}_{\mathcal{Z}'}) + 1 \geq 3$ and, by lemma 2, we obtain $H^1(X, \mathcal{I}(Z')) = 0$. Consider now the exact sequence of sheaves

$$0 \to \mathcal{I}(Z') \to \mathcal{O}_X \to \mathcal{O}_{Z'} \to 0$$

Since $H^1(X, \mathcal{I}(Z')) = 0$ and Z' is Stein, we deduce from the long exact sequence of cohomology that $H^1(X, \mathcal{O}_X) = 0$. ■

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