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Coefficient Inequalities for Certain Classes of Analytic and Univalent Functions in the Unit Disk

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Abstract

Let A denote the class of functions analytic in the unit disk $E = \{z : |z| < 1\}$ and normalized with $f(0) = 0$, $f'(0) - 1 = 0$. Recently, Hayami et al. [1] discussed some interesting results on coefficient inequalities for functions $f(z) \in A$ which are starlike, convex and λ - spiral of order α. The object of the present paper is to extend this results to the class $S_n(\alpha, \beta)$ defined by Salagean derivative operator to derive some interesting coefficient inequalities and discuss their consequences. Also, relevant connections of our results with some of the existing ones are discussed.

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1. Introduction

Let A denote the class of function $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$ and normalized with $f(0) = 0$ and $f'(0) - 1 = 0$.

Furthermore, we let Ω denote the class of function $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \tag{1.2}$$

which are analytic in E and satisfies $Rep(z) > 0$.

Let $S \subset A$ be the class of functions univalent in $E = \{z : |z| < 1\}$. Moreover, the normalized analytic function defined by (1.1) is said to be starlike, convex, close - to convex with the provision that the geometric quantities

$$\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, f'(z)$$

are respectively belong to the family Ω.

In this work the authors wish to define the following:

Definition 1.1. A function $f(z) \in A$ is said to be in the class $S_n(\alpha, \beta)$ if and only if

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) > \alpha \left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| + \beta \tag{1.3}$$

where $\alpha \geq 0, 0 \leq \beta < 1, z \in E$ and D^n in this work is the Salagean derivative operator defined as

$$D^0 f(z) = f(z), D^1 f(z) = zf'(z), \ldots, D^n f(z) = z(D^{n-1}f(z))' \tag{1.4}$$

For clearity purpose we give

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \;\; [2, 3, 4]$$

Also we shall need the following lemmas in our present investigation.

Lemma 1.2. [1] A function $p(z) \in E$ satisfy the following condition $Rep(z) > 0, z \in E$ if and only if $p(z) \neq \dfrac{\psi - 1}{\psi + 1}, (z \in E, \psi \in C, |\psi| = 1)$.

2. Coefficient inequalities for the class $S_n(\alpha, \beta)$

For our main results we first derive the following:

Lemma 2.1. A function $f(z) \in A$ is in the class $S_n(\alpha, \beta)$ if and only if

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0 \tag{2.1}$$

where

$$A_k = \frac{k^n[k+1-2\beta-\alpha(k-1)+(1-\alpha)(k-1)\psi]}{2(1-\beta)}a_k$$

$0 \le \beta < 1$ and $n \in N_0 = \{0, 1, 2, \ldots, \}, \alpha \ge 0$.

Proof. Let us set

$$p(z) = \frac{\frac{(1-\alpha)D^{n+1}f(z)}{D^n f(z)} - (\beta-\alpha)}{1-\beta} \tag{2.2}$$

We find that $p(z) \in \Omega$ and $Rep(z) > 0, z \in E$. By applying Lemma 1.2, we have

$$\frac{\frac{(1-\alpha)D^{n+1}f(z)}{D^n f(z)} - (\beta-\alpha)}{1-\beta} \neq \frac{\psi-1}{\psi+1} \tag{2.3}$$

which readily yields

$$\frac{(1-\alpha)D^{n+1}f(z) - (\beta-\alpha)D^n f(z)}{(1-\beta)D^n f(z)} \neq \frac{\psi-1}{\psi+1} \tag{2.4}$$

$(f(z) \in S_n(\alpha, \beta), z \in E, \psi \in C, |\psi| = 1$.

Thus we find that

$$\frac{(1-\beta)+(1-\alpha)\sum_{k=2}^{\infty} k^{n+1}a_k z^{k-1} - (\beta-\alpha)\sum_{k=2}^{\infty} k^n a_k z^{k-1}}{(1-\beta)+(1-\beta)\sum_{k=2}^{\infty} k^n a_k z^{k-1}} \neq \frac{\psi-1}{\psi+1} \tag{2.5}$$

which readily gives

$$1 + \sum_{k=2}^{\infty} \frac{k^n[k+1-2\beta-\alpha(k-1)+(1-\alpha)(k-1)\psi]a_k z^{k-1}}{2(1-\beta)} \neq 0 \tag{2.6}$$

which complete the proof of Lemma 2.1. ∎

On setting $\alpha = 0$ in Lemma 2.1 we have

Corollary 2.2. A function $f(z) \in A$ is in the class $S_n(0, \beta)$ if and only if

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0$$

where

$$A_k = \frac{k^n(k+1-2\beta+(k-1)\psi)}{2(1-\beta)}a_k$$

which shows that $f(z) \in S_n(0, \beta) \equiv S_n(\beta)$ see [2].

On setting $n = 0, 0 \leq \beta < 1, \alpha = 0$ in Lemma 2.1 we have

Corollary 2.3. A function $f(z) \in A$ is in the class $S_0(0, \beta)$ if and only if

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0$$

where

$$A_k = \frac{k + 1 - 2\beta + (k-1)\psi}{2(1-\beta)} a_k$$

$f(z) \in S^*(0, \beta) \equiv S^*(\beta)$ which is Lemma 2.1 of [1].

Also, putting $n = 1$, and $\alpha = 0, 0 \leq \beta < 1$ in Lemma 2.1 we have

Corollary 2.4. A function $f(z) \in A$ is in the class $S_1(0, \beta)$ if and only if

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0$$

where

$$A_k = \frac{k(k + 1 - 2\beta + (k-1)\psi)}{2(1-\beta)} a_k$$

$f(z)$ is convex.

On setting $\alpha = 0, \beta = \frac{1}{2}$ in Lemma 2.1 we have

Corollary 2.5. A function $f(z) \in A$ is in the class $S_n\left(0, \frac{1}{2}\right)$ if and only if

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0$$

where

$$A_k = k^n(k + (k-1)\psi) a_k$$

In view of lemma 2.1 we state and proof the following.

Theorem 2.6. If $f(z) \in A$ satisfies following inequality

$$\begin{aligned} &\sum_{k=2}^{\infty} \left(\left| \sum_{m=1}^{k} \left(\sum_{j=1}^{m} (-1)^{m-j} j^n (j + 1 - 2\beta - \alpha(j-1)) \binom{\rho}{m-j} T_j \right) (\gamma_{k-m}) \right| \right. \\ &\quad \left. + \left| \sum_{m=1}^{\infty} \left(\sum_{j=1}^{m} (-1)^{m-j} j^n (1-\alpha)(j-1) \binom{\rho}{m-j} T_j \right) \binom{\gamma}{k-m} \right| \right) \\ &\quad \leq 2(1-\beta) \end{aligned} \tag{2.7}$$

$0 \leq \beta < 1, \alpha \geq 0, \rho, \gamma \in R$ then $f(z) \in S_n(\alpha, \beta)$.

Proof. First we note that

$$(1-z)^\rho \neq 0 \; and \; (1+z)^\gamma \neq 0, \;\; (z \in E, \rho, \gamma \in R)$$

Hence, if the following inequality

$$\left(1 + \sum_{k=2}^{\infty} A_k z^{k-1}\right) (1-z)^\rho \, (1+z)^\gamma \neq 0 \; (z \in E, \rho, \gamma \in R)$$

holds true, then we have that

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0$$

which is the relation (2.1) of Lemma 2.1 It is easy to see that (2.1) is equivalent to

$$\left(1 + \sum_{k=2}^{\infty} A_k z^{k-1}\right) \left(\sum_{k=0}^{\infty} (-1)^k b_k z^k\right) \left(\sum_{k=0}^{\infty} c_k z^k\right) \neq 0 \tag{2.8}$$

where for convinience

$$b_k = \binom{\rho}{k+1}, \;\; c_k = \binom{\gamma}{k+1}$$

Considering the Cauchy product of the first two factors of (2.8), (2.8) can be written as

$$\left(1 + \sum_{k=2}^{\infty} H_k z^{k-1}\right) \left(\sum_{k=0}^{\infty} c_k z^{k-1}\right) \neq 0 \tag{2.9}$$

where

$$H_k = \sum_{j=1}^{k} (-1)^{k-j} A_j b_{k-j}.$$

Furthermore, by applying the same Cauchy product method in (2.9) we obtain

$$1 + \sum_{k=2}^{\infty} \left(\sum_{m=1}^{k} H_m c_{k-m}\right) z^{k-1}, \;\; z \in E$$

or written equivalently as

$$1 + \sum_{k=2}^{\infty} \left[\sum_{m=1}^{k} \left(\sum_{j=1}^{m} (-1)^{m-j} A_j b_{m-j}\right) c_{k-m}\right] z^{k-1} \neq 0, \;\; z \in E$$

That is if $f(z) \in A$ satisfies the following inequality

$$\sum_{k=2}^{\infty}\left|\sum_{m=1}^{k}\left(\sum_{j=1}^{m}(-1)^{m-j}A_j b_{m-j}\right)c_{k-m}\right| \leq 1$$

that is, if

$$\begin{aligned}
&\frac{1}{2(1-\beta)}\sum_{k=2}^{\infty}\left|\sum_{m=1}^{k}\left(\sum_{j=1}^{m}(-1)^{m-j}j^n[(j+1-2\beta-\alpha(j-1))\right.\right. \\
&\quad \left.\left.+(1-\alpha)(j-1)\psi]T_j b_{m-j}\right)c_{k-m}\right|. \\
&\leq \frac{1}{2(1-\alpha)}\sum_{k=2}^{\infty}\left(\left|\sum_{m=1}^{k}\left[\sum_{j=1}^{m}(-1)^{m-j}j^n(j+1-2\beta-\alpha(j-1))T_j b_{m-j}\right]c_{k-m}\right|\right. \\
&\quad \left.+|\psi|\left|\sum_{m=1}^{\infty}\left[\sum_{j=1}^{m}(-1)^{m-j}j^n(1-\alpha)(j-1)T_j b_{m-j}\right]c_{k-m}\right|\right) \\
&\leq 1
\end{aligned}$$

$0 \leq \beta < 1, \alpha \geq 0, \psi \in C, |\psi| = 1$ then $f(z) \in S_n(\alpha, \beta)$. This complete the proof of Theorem 2.6. ■

On setting $\alpha = 0$, in Theorem 2.6, we have the following

Corollary 2.7. If $f(z) \in A$ satisfies the following condition

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left(\left|\sum_{m=1}^{k}\left(\sum_{j=1}^{m}(-1)^{m-j}j^n(j+1-2\beta)\binom{\rho}{m-j}T_j\right)\binom{\gamma}{k-m}\right|\right. \\
&\quad \left.+\left|\sum_{m=1}^{\infty}\left(\sum_{j=1}^{m}(-1)^{m-j}j^n(j-1)\binom{\rho}{m-j}T^j\right)\binom{\gamma}{k-m}\right|\right) \\
&\leq 2(1-\beta)
\end{aligned} \tag{2.10}$$

then $f(z) \in S_n(0, \beta)$.

On setting $n = 0, \alpha = 0$ and $0 \leq \beta < 1$ in Theorem 2.6, we have the following

Corollary 2.8. If $f(z) \in A$ satisfies the following condition

$$\begin{aligned}\sum_{k=2}^{\infty}\Bigg(\Bigg|\sum_{m=1}^{k}\Bigg(\sum_{j=1}^{m}(-1)^{m-j}(j+1-2\beta)\binom{\rho}{m-j}T_j\Bigg)\binom{\gamma}{k-m}\Bigg|\\+\Bigg|\sum_{m=1}^{\infty}\Bigg(\sum_{j=1}^{m}(-1)^{m-j}(j-1)\binom{\rho}{m-j}T^j\Bigg)\binom{\gamma}{k-m}\Bigg|\Bigg)\\ \leq 2(1-\beta)\end{aligned} \tag{2.11}$$

then $f(z) \in S_0(0, \beta)$ and it is starlike of order β which is the Theorem 1 in [1].

Setting $n = 0, \alpha = 0$ and $\beta = 0$ in Theorem 2.6 we have

Corollary 2.9. If $f(z) \in A$ satisfies the following condition

$$\begin{aligned}\sum_{k=2}^{\infty}\Bigg(\Bigg|\sum_{m=1}^{k}\Bigg(\sum_{j=1}^{m}(-1)^{m-j}(j+1)\binom{\rho}{m-j}T_j\Bigg)\binom{\gamma}{k-m}\Bigg|\\+\Bigg|\sum_{m=1}^{\infty}\Bigg(\sum_{j=1}^{m}(-1)^{m-j}(j-1)\binom{\rho}{m-j}T_j\Bigg)\binom{\gamma}{k-m}\Bigg|\Bigg)\\ \leq 2,\end{aligned} \tag{2.12}$$

$\rho, \gamma \in R$ then $f(z) \in S^*(0, 0) \equiv S^*$ which is Corollary 2.2 [1].

Setting $n = 0, \alpha = 0$ and $\beta = \dfrac{1}{2}$ we have

Corollary 2.10. If $f(z) \in A$ satisfies the following condition

$$\begin{aligned}\sum_{k=2}^{\infty}\Bigg(\Bigg|\sum_{m=1}^{k}\Bigg(\sum_{j=1}^{m}(-1)^{m-j}j\binom{\rho}{m-j}T_j\Bigg)\binom{\gamma}{k-m}\Bigg|\\+\Bigg|\sum_{m=1}^{\infty}\Bigg(\sum_{j=1}^{m}(-1)^{m-j}(j-1)\binom{\rho}{m-j}T^j\Bigg)\binom{\gamma}{k-m}\Bigg|\Bigg)\\ \leq 1\end{aligned} \tag{2.13}$$

or equivalent to

$$\sum_{k=2}^{\infty} k(k-1)a_k \leq 1$$

Setting $n = 1, \alpha = 0$ and $0 \leq \beta < 1$ in Theorem 2.6 we have

Corollary 2.11. If $f(z) \in A$ satisfies the following condition

$$\sum_{k=2}^{\infty}\left(\left|\sum_{m=1}^{k}\left(\sum_{j=1}^{m}(-1)^{m-j}j(j+1-2\beta)\binom{\rho}{m-j}T_j\right)\binom{\gamma}{k-m}\right| + \left|\sum_{m=1}^{\infty}\left(\sum_{j=1}^{m}(-1)^{m-j}j(j-1)\binom{\rho}{m-j}T^j\right)\binom{\gamma}{k-m}\right|\right) \le 2(1-\beta) \tag{2.14}$$

then $f(z) \in S(0,\beta)$, $\rho, \gamma \in R$ and $f(z)$ is convex which is Theorem 2 of [1].

3. Coefficient inequality for functions in the class $S_n(\theta, \alpha, \beta)$

Here we consider the subclass $S_n(\theta,\alpha,\beta)$ of A which consist of function $f(z) \in A$ if and only if the following condition holds true

$$Re\left[e^{i\theta}\left(\frac{D^{n+1}f(z)}{D^n f(z)} - \left(\frac{\beta-\alpha}{1-\alpha}\right)\right)\right] > 0, \ \left(z \in E, 0 \le \alpha < 1, -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right) \tag{3.1}$$

For $f(z) \in S_n(\theta,\alpha,\beta)$, we first derive the following

Lemma 3.1. A function $f(z) \in A$ is in the class $S_n(\theta,\alpha,\beta)$ if and only if

$$1 + \sum_{k=2}^{\infty} L_k z^{k-1} \neq 0 \tag{3.2}$$

where

$$L_k = \frac{k^n[(k-1)(1-\alpha) - 2(1-\beta)e^{-i\theta}cos\theta + (1-\alpha)(k-1)\psi]}{2(1-\beta)e^{-i\theta}cos\theta}a_k$$

Proof. Set

$$p(z) = \frac{e^{i\theta}\left(\frac{(1-\alpha)D^{n+1}f(z)}{D^n f(z)} - (\beta-\alpha)\right) - i(1-\beta)sin\theta}{(1-\beta)cos\theta} \tag{3.3}$$

we can easily see that $p(z) \in \Omega$ and $Rep(z) > 0$, $z \in E$. It does follow from Lemma 1.2 that

$$\frac{e^{i\theta}(1-\alpha)(D^{n+1}f(z) - (\beta-\alpha)D^n f(z)) - i(1-\beta)sin\theta D^n f(z)}{(1-\beta)cos\theta D^n f(z)} \neq \frac{\psi-1}{\psi+1} \tag{3.4}$$

which after some simple computation yields

$$1 + \sum_{k=2}^{\infty} k^n \left(\frac{(k-1)(1-\alpha) + 2(1-\beta)e^{-i\theta}cos\theta + (1-\alpha)(k-1)\psi}{2(1-\beta)e^{-i\theta}cos\theta} \right) a_k z^{k-1} \neq 0 \tag{3.5}$$

$0 \le \alpha < 1, -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}, n - (0, 1, 2, \ldots)$ which complete the proof of lemma 3.1.■

In view of lemma 3.1 we state and proof the following

Theorem 3.2. If $f(z) \in A$ satisfies the following inequality

$$\sum_{k=2}^{\infty} \left(\left| \sum_{m=1}^{k} \left[\sum_{j=1}^{m} (-1)^{m-j} j^n ((j-1)(1-\alpha) - 2(1-\beta)e^{-2i\theta}) cos\theta \binom{\rho}{m-j} T_j \right] \binom{\gamma}{k-m} \right| + \left| \sum_{m=1}^{\infty} \left(\sum_{j=1}^{m} (-1)^{m-j} j^n (1-\alpha)(j-1) \binom{\rho}{m-j} T_j \right) \binom{\gamma}{k-m} \right| \right) \le 2(1-\alpha)cos\theta \tag{3.6}$$

$0 \le \alpha < 1, -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}, n = 0, 1, 2, \ldots,$ then $f(z) \in S_n(\theta, \alpha)$.

Proof. The proof follows the same method as in Theorem 2.6.

When $n = 0, \rho - 1 = \gamma = 0$ or $n = 1, n, \rho = \gamma = 1$ or $\rho - 2 = \gamma = 0$. Theorem 3.2 yields some existing results and new interesting ones.

At $\theta = 0$ Theorem 3.2 implies Theorem 2.6. Also, putting $\theta = 0, \alpha = 0$ in Theorem 3.2 some existing resuls can be obtained. ■

4. Conclusions

With various choices of $\alpha, \beta, \rho, \gamma, \theta$ and n many existing results can be derived and new ones can be obtained.

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