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On the Operator $\otimes^k$ Related to Nonlinear Heat Equation and its Spectrum

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Abstract

In this paper, we study the nonlinear equation of the form

$$\frac{\partial}{\partial t}u(x,t)-c^2(-\otimes^k u(x,t)=f(x,t,u(x,t))$$

where $\otimes^k$ is the operator iterated k-times, defined by

$$\otimes^k=\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3-\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k$$

where $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n, u(x,t)$ is an unknown for $(x,t)=(x_1,x_2,...,x_n,t)\in\mathbb{R}^n\times(0,\infty), k$ is a positive integer and c is a positive constant, f is the given function in nonlinear form depending on x,t and $u(x,t)$. On suitable conditions for f,p,q,k and the spectrum, we obtain the unique solution $u(x,t)$ of such equation. Moreover, if we put $q=0,k=1,$ we obtain the solution of non-linear heat equation.

Keywords: Diamond operator, Ultra-hyperbolic, Tempered distribution, Fourier transform.

Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t}u(x,t)=c^2\Delta u(x,t) \tag{1.1}$$

with the initial condition

$$u(x,0=f(x)$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $(x,t) = (x_1, x_2, ..., x_n, t) \in \mathbb{R}^n \times (0, \infty)$, we obtain

$$u(x,t) = \frac{1}{(4c^2 \pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4c^2 t} \right) f(y) dy \tag{1.2}$$

as the solution of (1.1). Now, (1.2) can be written $u(x,t) = E(x,t) * f(x)$ whre

$$E(x,t) = \frac{1}{(4c^2 \pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4c^2 t} \right). \tag{1.3}$$

$E(x,t)$ is called the heat kernel, where $|x|^2 = x_1^2 + x_2^2 + ... + x_n^2$ and $t > 0$ see [1, p208-209].

In 1996, A. Kananthai [2] has introduced the Diamond operator $\lozenge$ defined by

$$\lozenge = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2, \ p+q=n$$

or $\lozenge$ can be written as then product of the operators in the form $\lozenge = \Delta \Box = \Box \Delta$ where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}$ is the ultra-hyperbolic. The Fourier transform of the Diamond operator also has been studied and the elementary solution of such operator, see [3].

Next, K. Nonlaopon and A. Kananthai (see [5]) study the equation

$$\frac{\partial^2}{\partial t} u(x,t) = c^2 \Box u(x,t)$$

In this paper, we study the nonlinear equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 (-\otimes)^k u(x,t) = f(x,t,u(x,t)) \tag{1.4}$$

The operator $\otimes^k$ can be expressed in the form

$$\otimes^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right]^k$$

Where

$$\begin{aligned} \otimes^k &= \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right)^k \\ &= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k \end{aligned}$$

$$\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2+\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)+\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k$$
$$= (\Box)^k\left(\triangle^2-\frac{1}{4}(\triangle+\Box)(\triangle-\Box)\right)^k$$
$$= \left(\frac{3}{4}\Diamond\triangle+\frac{1}{4}\Box^3\right)^k$$

where

$$\triangle = \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_n^2}$$
$$\Box = \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\frac{\partial^2}{\partial x_{p+2}^2}-\cdots-\frac{\partial^2}{\partial x_{p+q}^2}$$
$$\Diamond = \left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_p^2}\right)^2-\left(\frac{\partial^2}{\partial x_{p+1}^2}+\frac{\partial^2}{\partial x_{p+2}^2}+\cdots+\frac{\partial^2}{\partial x_{p+q}^2}\right)^2$$

which is in the form of nonlinear heat equation. We consider the equation (1.4) with the following conditions on u and f as follows.

1. $u(x,t)\in C^{(6k)}(\mathbb{R}^n)$ for any $t>0$ where $C^{(6k)}(\mathbb{R}^n)$ is the space of continuous function with 6k-derivatives.
2. f satisfies the Lipchitz condition,

$$|f(x,t,u)-f(x,t,w)|\le A|u-w|$$

where A is constant with $0<A<1$.

3. $\int_0^\infty\int_{\mathbb{R}^n}|f(x,t,u(x,t)|dxdt<\infty$ for $x=(x_1,x_2,...,x_n)\in\mathbb{R}^n, 0<1<\infty$ and $u(x,t)$ is continuous function on $\mathbb{R}^n\times(0,\infty)$.

Under such conditions of f and u and for the spectrum of $E(x,t)$, we obtain the convolution

$$u(x,t)=E(x,t)*f(x,t,u(x,t))$$

as a unique solution of (1.4) where $E(x,t)$ is an elementary solution of (1.4).

Preliminaries

Definition 2.1 Let $f(x)\in L_1(\mathbb{R}^n)$ - the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f(\xi)}=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{-i(\xi,x)}f(x)dx \tag{2.1}$$

where $\xi=(\xi_1,\xi_2,...,\xi_n), x=(x_1,x_2,...,x_n)\in\mathbb{R}^n, (\xi,x)=\xi_1x_1+\xi_2x_2+...+\xi_nx_n$ is the usual ineer product in $\mathbb{R}^n$ and $dx=dx_1dx_2...dx_n$. Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f(\xi)} d\xi. \tag{2.2}$$

If f is a distribution with compact supports by [6], Theorem 7.4-3, p. 187 Eq. (2.1) can be written as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i(\xi,x)} \right\rangle. \tag{2.3}$$

Definition 2.2 The spectrum of the kernel $E(x,t)$ defined by (2.6) is the bounded support of the Fourier transform $\widehat{E(\xi,t)}$ for any fixed $t > 0$.

Definition 2.3 Let $\xi = (\xi_1, \xi_2, ..., \xi_n)$ be a point in $\mathbb{R}^n$ and write

$$u = \xi_1^2 + \xi_2^2 + ... + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - ... - \xi_{p+q}^2, \; p + q = n.$$

Denote by $\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1 > 0 \text{ and } u > 0\}$ the set of an interior of the forward cone and denote by $\overline{\Gamma}_+$ the closure of $\overline{\Gamma}_+$. Let Ω be the spectrum of $E(x,t)$ for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E(\xi,t)}$ be the Fourier transform of $E(x,t)$ and define

$$\widehat{E(\xi,t)} = \begin{cases} \dfrac{1}{(2\pi)^{n/2}} \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right)^k \right], & \text{for } \xi \in \Gamma_+ \\ 0, & \text{for } \xi \notin \Gamma_+ \end{cases} \tag{2.4}$$

Lemma 2.1 The Fourier transform of $(-\otimes)^k \delta$)

$$\mathcal{F}(-\otimes)^k \delta = \frac{(-1)^{4k}}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2 \right)^3 - \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2 \right)^3 \right]^k$$

where F is the Fourier transform defined by Eq. (2.1) and if the norm of ξ is given by $\|\xi\| = (\xi_1^2 + \xi_2^2 + ... + \xi_n^2)^{1/2}$ then

$$\mathcal{F}(-\otimes)^k \delta \leq \frac{3}{(2\pi)^{n/2}} \|\xi\|^{6k}$$

that is $F(-\otimes)^k$ is bounded and continuous on the space S' of the tempered distribution.

Moreover, by Eq. (2.2)

$$(-\otimes)^k \delta = \mathcal{F}^{-1} \frac{1}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2 \right)^3 - \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2 \right)^3 \right]^k$$

Proof. By Eq. (2.3)

$$
\begin{aligned}
\mathcal{F}(-\otimes)^k\delta &= \frac{1}{(2\pi)^{n/2}}\left\langle(-\otimes)^k\delta, e^{-i(\xi\cdot x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\otimes)^k e^{-i(\xi\cdot x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\otimes)^{k-1}(-\odot)e^{-i(\xi\cdot x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\odot)^{k-1}\left(-\frac{3}{4}\Diamond\triangle-\frac{1}{4}\Box^3\right)e^{-i(\xi\cdot x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\odot)^{k-1}\left(-\frac{3}{4}\Diamond\triangle\right)e^{-i(\xi\cdot x)}\right\rangle+\frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\odot)^{k-1}\left(-\frac{1}{4}\Box^3\right)e^{-i(\xi\cdot x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\otimes)^{k-1}\frac{3}{4}(-1)^3\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right](-1)\left(\sum_{i=1}^{n}\xi_i^2\right)e^{-i(\xi\cdot x)}\right\rangle \\
&\quad +\frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\otimes)^{k-1}\frac{1}{4}(-1)^4\left[\left(\sum_{i=1}^{p}\xi_i^2\right)-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)\right]^3 e^{-i(\xi\cdot x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\otimes)^{k-1}\frac{3}{4}(-1)^4\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right]\left(\sum_{i=1}^{n}\xi_i^2\right)e^{-i(\xi\cdot x)}\right\rangle \\
&\quad +\frac{1}{2\pi)^{n/2}}\left\langle\delta, (-\odot)^{k-1}\left(\frac{1}{4}(-1)^4\left[\left(\sum_{i=1}^{p}\xi_i^2\right)-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)\right]^3\right)e^{-i(\xi\cdot x)}\right\rangle \\
&= \frac{(-1)^4}{(2\pi)^{n/2}}\left\langle\delta, (-\otimes)^{k-1}\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]e^{-i(\xi\cdot x)}\right\rangle
\end{aligned}
$$

By keeping on operator $(-\otimes)$ with k-1 times, we obtain

$$\mathcal{F}(-\otimes)^k\delta=\frac{(-1)^{4k}}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3-\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right]^k$$

Now,

$$
\begin{aligned}
|\mathcal{F}(-\otimes)^k\delta| &= \frac{1}{(2\pi)^{n/2}}\left|\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3-\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right|^k \\
&\le \frac{1}{(2\pi)^{n/2}}\left|\xi_1^2+\ldots+\xi_n^2\right|^k\left|\left(\xi_1^2+\ldots+\xi_n^2\right)^2+\left(\xi_1^2+\ldots+\xi_n^2\right)^2+\left(\xi_1^2+\ldots+\xi_n^2\right)^2\right|^k \\
&\le \frac{3}{(2\pi)^{n/2}}\|\xi\|^{6k}
\end{aligned}
$$

where $\|\xi\|=(\xi_1^2+\xi_2^2+...+\xi_n^2)^{1/2}, \xi_i(i=1,2,...,n)\in\mathbb{R}$. Hence we obtain $F\otimes\delta$ is bounded and continuous on the space S' of the tempered distribution.

Since F is 1-1 transformation from the space S' of the tempered distribution to the real space $\mathbb{R}$, then by (2.2)

$$\otimes\delta=\mathcal{F}^{-1}\frac{1}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3-\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right].$$

That completes the proof.

Lemma 2.2 Let L be the operator defined by

$$L=\frac{\partial}{\partial t}-c^2(-\otimes)^k \tag{2.5}$$

where $\otimes^k$ is the operator iterated k-times defined by

$$\otimes^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k,$$

$p+q=n$ is the dimension of $\mathbb{R}^n, (x_1, x_2, ..., x_n) \in \mathbb{R}^n, t \in (0,\infty), k$ is a positive integer and c is the positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k + i(\xi,x)\right]d\xi \tag{2.6}$$

as the elementary solution of (1.4) in the spectrum $\Omega \subset \mathbb{R}^n$ for $t>0$, where $\sum_{j=p+1}^{p+q}\xi_j^2 > \sum_{i=1}^{p}\xi_i^2$.

Proof. Let $LE(x,t) = \delta(x,t)$ where $E(x,t)$ is the kernel or the elementary solution of the operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t}E(x,t) - c^2(-\otimes)^k E(x,t) = \delta(x)\delta(t)$$

take the Fourier transform defined by (2.1) to both sides of the equation

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} - c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]^k \widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t).$$

Thus

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}}\exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right]$$

where $H(t)$ is the Heaviside function. Since $H(t)=1$ for $t>0$,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right],$$

so we have

$$E(\xi,t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{E(\xi,t)}d\xi.$$

By (2.3)

$$E(\xi,t) = \frac{1}{(2\pi)^{n/2}}\int_\Omega e^{i(\xi,x)}\widehat{E(\xi,t)}d\xi$$

Where Ω is the spectrum of $E(x,t)$. Thus

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k + i(\xi,x)\right]d\xi.$$

for $t>0$.

Definition 2.4 We can extend $E(x,t)$ to $\mathbb{R}^n \times \mathbb{R}$ by setting

$$E(x,t)=\begin{cases}\dfrac{1}{(2\pi)^n}\displaystyle\int_\Omega \exp\left[c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right], & \text{for } t>0\\ 0, & \text{for } t\le 0.\end{cases}$$

Lemma 2.3 (The properties of $E(x,t)$

The kernel $E(x,t)$ defined by (2.6) have the following properties

1. $E(x,t)\in C^\infty$ - the space of continuous function for $x\in\mathbb{R}^n, t>0$ with infinitely detterentiable.
2. $\left(\dfrac{\partial}{\partial t}-c^2(-\otimes)^k\right)E(x,t)=0$ for $t>0$.
3. $|E(x,t)|\le \dfrac{2^{2-n}}{\pi^{n/2}}\dfrac{M(t)}{\pi^{1/2}\Gamma(p/2)\Gamma(q/2)}$ for $t>0$ where $M(t)$ is a function of t in the spectrum and Γ denote the Gamma function. Thus $E(x,t)$ is bounded for any fixed $t>0$.
4. $\lim\limits_{t\to 0}E(x,t)=\delta.$

Proof. (1) From (2.6)

$$\frac{\partial^n}{\partial x^n}E(x,t)=\frac{1}{(2\pi)^n}\int_\Omega \frac{\partial^n}{\partial x^n}\exp\left[c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k+i(\xi,x)\right]d\xi.$$

Thus $E(x,t)\in C^\infty$ for $x\in\mathbb{R}^n, t>0.$

(2) By computing directly, we obtain

$$\left(\frac{\partial}{\partial t}-c^2(-\otimes)^k\right)E(x,t)=0.$$

(3) We have

$$E(x,t)=\frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k+i(\xi,x)\right]d\xi.$$

Thus

$$|E(x,t)|\le\frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right]d\xi.$$

By chaning to bipolar coordinates

$\xi_1=rw_1,\xi_2,=rw_2,...,\xi_p=rw_p$ and $\xi_{p+1}=sw_{p+1},\xi_{p+2}=sw_{p+2},...,\xi_{p+q},=sw_{p+q}$

where $\sum\limits_{i=1}^{p}w_i^2=1$ and $\sum\limits_{j=p+1}^{p+q}w_j^2=1$

Thus

$$\sum_{i=1}^{p} w_i^2 = 1 \quad \text{and} \quad \sum_{j=p+1}^{p+q} w_j^2 = 1$$

where $d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Since $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ and suppose $0 \le r \le R$ and $0 \le s \le L$ where R and L are constants. Thus we obtain

$$\begin{aligned} |E(x,t)| &\le \frac{\Omega_p\Omega_q}{(2\pi)^n}\int_0^R\int_0^L \exp\left[c^2t\left(r^6-s^6\right)^k\right] r^{p-1}s^{q-1}drds \\ &= \frac{\Omega_p\Omega_q}{(2\pi)^n}M(t) \quad \text{for any fixed } t>0 \text{ in the spectrum } \Omega \\ &= \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(p/2)\Gamma(q/2)} \end{aligned} \tag{2.7}$$

where $M(t) = \int_0^R \int_0^L \exp\left[c^2t\left(r^6-s^6\right)^6\right] r^{p-1}s^{q-1}drds$ is a function for $t > 0, \Omega_p = \dfrac{2\pi^{p/2}}{\Gamma(p/2)}$ and $\Omega_q = \dfrac{2\pi^{q/2}}{\Gamma(q/2)}$. Thus for any fixed $t > 0$, $E(x,t)$ is bounded.

(4) From (2.5),

$$\lim_{t\to 0} E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega e^{i(\xi,x)}d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\xi,x)}d\xi = \delta(x),$$

for $x \in \mathbb{R}^n$, see[4, p. 396, Eq. (10.2.19b)].

Main Results

Theorem 3.1 Given the nonlinear equation

$$\frac{\partial}{\partial t}u(x,t) - c^2(-\otimes)^k u(x,t) = f(x,t,u(x,t)) \tag{3.1}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty), k$ is a positive number and with the following conditions on u and f as follows

1. $u(x,t) \in C^{(6k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(6k)}(\mathbb{R}^n)$ is the space of continuous function with 6k-derivative.
2. f satisfies the Lipchitz condition,

$$|f(x,t,u) - f(x,t,w)| \le A|u-w|$$

where A is constant with $0 < A < 1$.

3. $\int_0^\infty \int_{\mathbb{R}^n} |f|x,t,u(x,t))|dxdt < \infty$ for $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, 0 < 1 < \infty$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0,\infty)$.

Then obtain the convolution

$$u(x,t) = E(x,t) * f(x,t,u(x,t)) \tag{3.2}$$

as a unique solution of (3.1) for $x \in \Omega$ where Ω is a compact subset of $\mathbb{R}^n$ and

$0 \le t \le T$ with T is constant and $E(x,t)$ is an elementary solution defined by (2.6) and also $u(x,t)$ is bounded for any fixed $t > 0$. In particular, if we put $k = 1$ and $p = 0$ in (3.1), then (3.1) reduces to the nonlinear hear equation

$$\frac{\partial}{\partial t}u(x,t) + c^2\Delta^3 u(x,t) = f(x,t,u(x,t))$$

which is relate to the heat equation.

Proof. Convolving both sides of (3.1) with $E(x,t)$, that is

$$E(x,t) * \left[\frac{\partial}{\partial t}u(x,t) - c^2(-\otimes^k u(x,t)\right] = E(x,t) * f(x,t,u(x,t))$$

or

$$\left[\frac{\partial}{\partial t}E(x,t) - c^2(-\otimes)^k E(x,t)\right] * u(x,t) = E(x,t) * f(x,t,u(x,t)),$$

so

$$\delta(x,t) * u(x,t) = E(x,t) * f(x,t,u(x,t)).$$

Thus

$$\begin{aligned} u(x,t) &= E(x,t) * f(x,t,u(x,t)) \\ &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} E(r,s) f(x-r,t-s,u(x-r,t-s))drds \end{aligned}$$

where $E(r,s)$ is given by definition (2.5). We next show that $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. We have

$$\begin{aligned} |u(x,t)| &\le \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |E(r,s)||f(x-r,t-s,u(x-r,t-s))|drds \\ &\le \frac{2^{2-n}NM(t)}{\pi^{n/2}\Gamma(p/2)\Gamma(q/2)} \qquad \text{by condition (3) and (2.6)} \end{aligned}$$

where $N = \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |f(x-r,t-s,u(x-r,t-s))|drds$. Thus $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. To show that $u(x,t)$ is unique. Now, we next to show that $u(x,t)$ is unique. Let $w(x,t)$ be another solution of (3.1), then

$$w(x,t) = E(x,t) * f(x,t,w(x,t))$$

for $(x,t) \in \Omega_0 \times (0,T]$ the compact subset of $\mathbb{R}^n \times [0,\infty)$ and $E(x,t)$ is defined by (2.6).

Now, define $\|u(x,t)\| = \sup_{\substack{x\in\Omega_0 \\ 0<t\le\Gamma}} |u(x,t)|$.

Now,

$$\begin{aligned} |u(x,t) - w(x,t)| &= |E(x,t) * f(x,t,u(x,t)) - E(x,t) * f(x,t,w(x,t))| \\ &\le \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |E(r,s)| \cdot |f(x-r,t-s,u(x-r,t-s)) \\ &\qquad - f(x-r,t-s,w(x-r,t-s))|drds \\ &\le A|E(r,s)|\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |u(x-r,t-s) - w(x-r,t-s)|drds \end{aligned}$$

by (2.6) and the condition (2) of the theorem. Now, for $(x,t)\in \Omega_0 \times (0,T]$ we have

$$\begin{aligned} |u-w| &\leq A|E(r,s)|\|u-w\| \int_0^T ds \int_{\Omega_0} dr \\ &= A|E(r,s)|TV(\Omega_0)\|u-w\| \end{aligned} \quad (3.3)$$

where $V(\Omega_0)$ is the volume of the surface on Ω_0.

Choose $A|E(r,s)|TV(\Omega_0)\leq 1$ or $A\leq \dfrac{1}{|E(r,s)|TV(\Omega_0)}$.

Thus from (3.3),

$$\|u-w\|\leq \alpha\|u-w\| \text{ where } \alpha = A|E(r,s)|TV(\Omega_0)\leq 1.$$

It follows that $\|u-w\|=0$, thus $u=w$.

That is the solution u of (3.1) is unique.

In particular, if we put $k=1$ and $q=0$ in (3.1), then (3.1) reduces to the nonlinear heat equation

$$\frac{\partial}{\partial t}u(x,t)+c^2\Delta^3 u(x,t) = f(x,t,u(x,t))$$

which has solution

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

where $E(x,t)$ is defined by (2.6) with $k=1$ and $q=0$.

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