

Pólya Theory for Orbiquotient Sets

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Abstract

Replacing the usual notion of quotient sets by the notion of orbiquotient sets we obtain a generalization of Pólya theory. The key ingredient of our extended theory is the definition of the orbicycle index polynomial which we compute in several examples. We apply our theory to the study of orbicycles on orbiquotient sets.

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1. Introduction

Assume that a finite group G acts on a finite set X . The quotient X/G of the action of G on X is a rich and subtle concept, traditionally $X/G = \{\bar{x} \mid x \in X\}$ where $\bar{x} = \{gx \mid x \in X\}$. In recent years it has proven convenient to modify this notion in various contexts. For example one may think of X/G as the groupoid whose set of objects is X and with morphisms given by $X/G(a, b) = \{g \in G \mid ga = b\}$ for $a, b \in X$. Following Connes [8] the groupoid X/G is studied with the methods of non-commutative geometry, i.e. looking at the convolution (incidence) algebra of X/G . Another approach to quotient sets became rather popular after Vafa and Witten introduced in [17] the so called stringy Euler numbers. In a nutshell they considered the Euler numbers of orbiquotient sets

$$X/{}^{orb}G = \bigsqcup_{\bar{g} \in C(G)} X^{\bar{g}}/Z(\bar{g}),$$

where $C(G)$ is the set of conjugacy classes of G , $X^g \subseteq X$ is the set of points fixed by $g \in G$, and $Z(g)$ is the centralizer of g in G . We will assume that a representative $g \in G$ has been chosen for each conjugacy class \bar{g} of G . Orbiquotient sets first appeared, see [2], in the context of equivariant K-theory in the works of Atiyah and Segal. The goal of this paper is to bring the notion of orbiquotient sets into combinatorial waters. Let us provide a combinatorial motivation for the study of orbiquotient sets inspired by an analogue topological construction given by Hirzebruch and Höfer in [14]. Consider the set of n -cycles in X/G , i.e., the set of maps $f : \mathbb{Z} \rightarrow X/G$ such that $f(k) = f(k + n)$ for $k \in \mathbb{Z}$. Suppose we want to lift f to a map $l : \mathbb{Z} \rightarrow X$ such that $\pi \circ l = f$, where $\pi : X \rightarrow X/G$ is the canonical projection. The lift l will not be unique, indeed if l is a lift then gl is another a lift; also we have that $l(k) = gl(k + n)$ for all $k \in \mathbb{Z}$ and some $g \in G$. Thus the set of lifts of n -periodic maps $\mathbb{Z} \rightarrow X/G$ may be identified with

$$\{l : \mathbb{Z} \rightarrow X \mid l(k) = gl(k + n) \text{ for } k \in \mathbb{Z} \text{ and some } g \in G\}/G.$$

Inside the later set sits $I(G, X)/G$ the set of constant maps, where

$$I(G, X) = \{(g, x) \in G \times X \mid gx = x\}$$

is the so called inertial set [15]. The group G acts on $I(G, X)$ as $k(g, x) = (kgk^{-1}, kx)$, and it is not hard to see that

$$I(G, X)/G = \bigsqcup_{\bar{g} \in C(G)} X^g/Z(g) = X^{orb}G.$$

In this paper we develop orbianalogues for two main results in elementary combinatorics, the orbit counting lemma and the Polya-Redfield theorem, see [4, 16]. We fix a commutative ring \mathbb{A} and consider the category of \mathbb{A} -weighted sets whose objects are pairs (X, f) where X is a finite set and $f : X \rightarrow \mathbb{A}$ is an arbitrary map called the weight of X . Morphisms between \mathbb{A} -weighted sets are weight preserving bijections. The cardinality $|X|_f$ of a weighted set (X, f) is given by

$$|X|_f = \sum_{x \in X} f(x).$$

A finite group G acts on (X, f) if G acts on X and $f(gx) = f(x)$ for all $g \in G, x \in X$. The Cauchy-Frobenius-Burnside orbit counting lemma gives us a way to compute $|X/G|_f$ as follows:

$$|X/G|_f = \frac{1}{|G|} \sum_{g \in G} |X^g|_f.$$

Suppose now that G is a group of permutations $G \subset S_m$. The cardinality of X^m/G is determined by the Pólya-Redfield theorem:

$$|X^m/G|_f = P_G(|X|_f, |X|_{f^2}, \dots, |X|_{f^m}),$$

where P_G is the cycle index polynomial of G given by

$$P_G(x_1, x_2, \dots, x_m) = \frac{1}{|G|} \sum_{g \in G} x_1^{c_1(g)} \dots x_m^{c_m(g)},$$

and $c_i(g)$ is the number of g -cycles of length i . If $X = [n]$ and $f(i) = x_i$ for $i \in X$, then directly from the definition of quotient sets we get that

$$|[n]^m/G|_f = \sum_{(i_1, \dots, i_n) \in \mathbb{N}^n} c_G(i_1, \dots, i_n) x_1^{i_1} \dots x_n^{i_n},$$

where $c_G(i_1, \dots, i_n)$ counts the colorations of $[m]$ with i_k elements of color $k \in [n]$, and two colorations are identified if they are linked by the action of G . The Pólya-Redfield theorem allows us to compute the coefficients $c_G(i_1, \dots, i_n)$ in a different way, namely we have that

$$|[n]^m/G|_f = P_G \left(\sum_{j=1}^n x_j, \sum_{j=1}^n x_j^2, \dots, \sum_{j=1}^n x_j^m \right).$$

The rest of this work is organized as follows. In Section 2 we provide an orbi-analogue of the orbit counting lemma. In Section 3 we provide an orbi-analogue of the Pólya-Redfield theorem in full generality, we shall see that lattice of partitions plays a fundamental role in our presentation. In the remaining sections we explicitly compute the orbicycle index polynomial for various groups in increasing order of difficulty. In Section 4 we consider the case of cyclic groups, and apply it to the study of orbicycles in orbiquotient sets. In Section 5 we consider the full symmetric group. In a rather dull fashion we may regard combinatorics as geometry in dimension zero. It is thus rather interesting when one can show that the zero dimensional combinatorial case determines the higher dimensional situation. A theorem of this sort is proved at the end of Section 5 which provides a strong motivation for the study of orbiquotient sets. In Section 6 we compute the orbicycle index polynomial for the dihedral groups.

2. Orbi-analogue of the orbit counting lemma

If $S \subseteq G$ and G acts on X , then we set $X^S = \{x \in X \mid gx = x \text{ for } g \in S\}$. Also we let $\langle g_1, \dots, g_n \rangle$ be the subgroup of G generated by $\{g_1, \dots, g_n\} \subset G$.

Definition 1. The orbiquotient of X by the action of G is the set given by

$$X/{}^{orb}G = \bigsqcup_{\bar{g} \in C(G)} X^g/Z(g).$$

The orbiquotient $X/{}^{orb}G$ is well defined up to canonical bijections. Indeed if $h = k g k^{-1}$ then the map $\psi : X^g \rightarrow X^h$ given by $\psi(x) = kx$ induces a bijection

$$\psi : X^g/Z(g) \rightarrow X^h/Z(h).$$

If G acts on a weighted set (X, f) then $X/{}^{orb}G$ is also weighted: X^g is weighted by $f|_{X^g}$ and $X^g/Z(g)$ is weighted by $f(\bar{x}) = f(x)$ for $\bar{x} \in X^g/Z(g)$. Our next result is the

orbi-analogue of the orbit counting lemma, let us first introduce a notation that will be used repeatedly

$$P(G) = \{(\bar{g}, h) \mid \bar{g} \in C(G) \text{ and } h \in Z(g)\}.$$

Theorem 2. If G acts on (X, f) , then the cardinality of $X/^{orb}G$ is given by

$$\left| X/^{orb}G \right|_f = \frac{1}{|G|} \sum_{(\bar{g}, h) \in P(G)} |\bar{g}| \left| X^{(g,h)} \right|_f.$$

The proof of this result is quite simple:

$$\begin{aligned} \left| X/^{orb}G \right|_f &= \sum_{\bar{g} \in C(G)} \left| X^{\bar{g}}/Z(g) \right|_f \\ &= \sum_{\bar{g} \in C(G)} \frac{|\bar{g}|}{|G|} \sum_{h \in Z(g)} \left| X^{\bar{g}} \cap X^h \right|_f \\ &= \frac{1}{|G|} \sum_{(\bar{g}, h) \in P(G)} |\bar{g}| \left| X^{(g,h)} \right|_f. \end{aligned}$$

3. Orbi-analogue of Pólya-Redfield theorem

Let $\text{Par}(X)$ be lattice of partitions of X . The minimal and maximal elements of $\text{Par}(X)$ are $\{\{x\} \mid x \in X\}$ and $\{X\}$, respectively. The joint $\pi \vee \rho$ of partitions π and ρ is defined by demanding that $i, j \in X$ belong to a block of $\pi \vee \rho$ if there exists a sequence $i = a_0, a_1, \dots, a_n = j$, such that for $0 \leq i \leq n - 1$ either a_i and a_{i+1} belong to a block in π , or a_i and a_{i+1} belong a block in ρ . The meet of partitions π and ρ is $\pi \wedge \rho = \{B \cap C \mid B \in \pi, C \in \rho, B \cap C \neq \emptyset\}$. Let the group G act on a set X with n -elements. Each $g \in G$ induces a partition $C(g)$ on X such that $C(g) = \bigsqcup_{i=1}^n C_i(g)$, where $C_i(g) = \{g\text{-cycles on } X \text{ of length } i\}$ for $1 \leq i \leq n$. We use the notation $c(g) = |C(g)|$ and $c_i(g) = |C_i(g)|$ for $1 \leq i \leq n$. If π is a partition of X we let $b_k(\pi)$ be the number of blocks of π of cardinality k .

Definition 3. The orbicycle index polynomial $P_G^{orb}(x_1, x_2, \dots) \in \mathbb{Q}[x_1, x_2, \dots]$ is given by

$$P_G^{orb}(x_1, x_2, \dots) = \frac{1}{|G|} \sum_{(\bar{g}, h) \in P(G)} |\bar{g}| x^{C(g) \vee C(h)},$$

where

$$x^{C(g) \vee C(h)} = \prod_{k \geq 1} x_k^{b_k(C(g) \vee C(h))}.$$

If $G \subseteq S_m$ then G acts on X^m . Suppose that $g, h \in G$ commute, then $i, j \in [m]$ belong to the same block of $C(g) \vee C(h)$ if and only if there exist $a, b \in \mathbb{Z}$ such that

$j = (g^a h^b)(i)$. It is easy to check that $f \in X^m$ is fixed by g and h if and only if f is constant on each block of $C(g) \vee C(h)$.

Theorem 4. Let (X, f) be an \mathbb{A} -weighted set and $G \subseteq S_m$. The cardinality of $X^m /^{orb} G$ is given by

$$|X^m /^{orb} G|_f = P_G^{orb}(|X|_f, |X|_{f^2}, \dots).$$

Proof.

$$\begin{aligned} |X^m /^{orb} G|_f &= \frac{1}{|G|} \sum_{(\bar{g}, h) \in P(G)} |\bar{g}| \left| (X^{[m]})^{\bar{g}} \cap (X^{[m]})^h \right|_{\tilde{f}} \\ &= \frac{1}{|G|} \sum_{(\bar{g}, h) \in P(G)} |\bar{g}| \sum_{\alpha: [m] \rightarrow X, \alpha \circ g = \alpha \circ h = \alpha} \prod_{x \in [m]} f(\alpha(x)) \\ &= \frac{1}{|G|} \sum_{(\bar{g}, h) \in P(G)} |\bar{g}| \sum_{\alpha: C(g) \vee C(h) \rightarrow X} \prod_{B \in C(g) \vee C(h)} \prod_{x \in B} f(\alpha(x)) \\ &= \frac{1}{|G|} \sum_{(\bar{g}, h) \in P(G)} |\bar{g}| \prod_{B \in C(g) \vee C(h)} \sum_{y \in X} f(y)^{|B|} \\ &= \frac{1}{|G|} \sum_{(\bar{g}, h) \in P(G)} |\bar{g}| \prod_{k \geq 1} |X|_{f^k}^{b_k(C(g) \vee C(h))} \\ &= P_G^{orb}(|X|_f, |X|_{f^2}, \dots). \end{aligned}$$

□

Let $X = [n]$ and $f(i) = x_i$, then one can check directly from the definition that

$$|[n]^m /^{orb} G|_f = \sum_{(i_1, \dots, i_n) \in \mathbb{N}^n} c_G^{orb}(i_1, \dots, i_n) x_1^{i_1} \dots x_n^{i_n},$$

where $c_G^{orb}(i_1, \dots, i_n)$ counts colorations c of $[m]$ with colors in $[n]$ such that:

- There are i_k elements in $[m]$ of color $k \in [n]$.
- c is g -invariant for some $\bar{g} \in C(G)$.
- Two g -invariant colorations are identified if they can be linked by the action of $Z(g)$.

The orbi-analogue of the Pólya-Redfield gives us another way to compute the coefficients $c_G^{orb}(i_1, \dots, i_n)$, namely we have that

$$|[n]^m /^{orb} G|_f = P_G^{orb} \left(\sum_{j=1}^n x_j, \sum_{j=1}^n x_j^2, \dots, \sum_{j=1}^n x_j^n \right).$$

4. Orbicycle index polynomial of \mathbb{Z}_n

Let \mathbb{N}_+ be the set of positive integers and let (x_1, x_2, \dots, x_k) be the greatest common divisor of $x_1, x_2, \dots, x_k \in \mathbb{N}^+$. The cyclic group with n -elements is denoted by \mathbb{Z}_n . For $n, k \in \mathbb{N}_+$ we define an equivalence relation on \mathbb{Z}_n^k as follows: x and y are equivalent if and only if $(x, n) = (y, n)$. It is easy to verify that $\mathbb{Z}_n^k = \bigsqcup_{d|n} \{x \in \mathbb{Z}_n^k \mid (x, n) = d\}$, and thus we have

$$n^k = \sum_{d|n} |\{x \in \mathbb{Z}_n^k \mid (x, n) = d\}|.$$

For $n \in \mathbb{N}_+$ the Jordan totient function J_k , see [1], is given by $J_k(n) = |\{x \in \mathbb{Z}_n^k \mid (x, n) = 1\}|$. For each $d|n$ we have $J_k\left(\frac{n}{d}\right) = |\{x \in \mathbb{Z}_n^k \mid (x, n) = d\}|$, therefore we get that $n^k = \sum_{d|n} J_k(d)$. By the Möbius inversion formula $J_k(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d^k$, thus $J_k(p^r) = p^{kr} - p^{k(r-1)}$, for p prime, and for arbitrary integer $n = p_1 \dots p_r$ we get

$$J_k(n) = n^k \left(1 - \frac{1}{p_1^k}\right) \dots \left(1 - \frac{1}{p_r^k}\right).$$

We shall need the following property, an easy consequence of the previous considerations, of the Jordan totient function:

$$\sum_{x \in \mathbb{Z}_n^k} f((x, n)) = \sum_{d|n} J_k(d) f(d),$$

for any $f : \{d : d \mid n\} \rightarrow \mathbb{A}$. Recall that if $x, y \in \mathbb{Z}_n \subseteq S_n$, then $|C(x) \vee C(y)| = (x, y, n)$, and all blocks in $C(x) \vee C(y)$ are of cardinality $\frac{n}{(n, x, y)}$. Indeed if $x \in \mathbb{Z}_n$, then $\frac{\mathbb{Z}_n}{(x)} \cong \mathbb{Z}_{(n, x)}$, since for $a, b \in \mathbb{Z}_n$ we have that $a = b$ in $\mathbb{Z}_n/(x)$ if and only if $a = b \pmod{(n, x)}$. Thus there are (n, x) blocks in $\mathbb{Z}_n/(x)$ all of them of cardinality $\frac{n}{(n, x)}$. Similarly if $x, y \in \mathbb{Z}_n$, then $a, b \in \mathbb{Z}_n$ are in the same block of $C(x) \vee C(y)$ if and only if there exist $r, s \in \mathbb{Z}$ such that $b = a + rx + sy \pmod n$, or equivalently $a = b \pmod{(n, x, y)}$.

Theorem 5.

$$P_{\mathbb{Z}_n}^{orb}(y_1, y_2, \dots, y_n) = \frac{1}{n} \sum_{d|n} J_2(d) y_d^{\frac{n}{d}}.$$

Proof.

$$\begin{aligned}
 P_{\mathbb{Z}_n}^{orb}(y_1, y_2, \dots, y_n) &= \frac{1}{n} \sum_{(\bar{g}, h) \in P(\mathbb{Z}_n)} |\bar{g}| \prod_{k \geq 1} y_k^{b_k(C(g) \vee C(h))} \\
 &= \frac{1}{n} \sum_{(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_n} \prod_{k \geq 1} y_k^{b_k(C(x) \vee C(y))} \\
 &= \frac{1}{n} \sum_{(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_n} y_{\frac{(x, y, n)}{(x, y, n)}}^{(x, y, n)} \\
 &= \frac{1}{n} \sum_{d|n} J_2(d) y_d^{\frac{n}{d}}.
 \end{aligned}$$

□

The coefficients $c_{\mathbb{Z}_n}(i_1, \dots, i_m)$ are computed in [13]. As a corollary of the previous result we obtain that

$$c_{\mathbb{Z}_n}^{orb}(i_1, \dots, i_m) = \frac{1}{n} \sum_{d|(i_1, \dots, i_m)} J_2(d) \frac{\left(\frac{i_1}{d} + \dots + \frac{i_m}{d}\right)!}{\frac{i_1}{d}! \dots \frac{i_m}{d}!}.$$

Let p be a prime number and $r \in \mathbb{N}^+$, necklaces without a clasp with p beads and r colors may be identified with the set $C_p[r] = [r]^p / \mathbb{Z}_p$. As explained in [18] we have that

$$|C_p[r]| = P_{\mathbb{Z}_p}(r, r, \dots) = \frac{1}{p} \sum_{d|p} \varphi(d) r^{\frac{p}{d}} = r + \frac{r^p - r}{p}.$$

Definition 6. The set $C_n^{orb}(X)$ of orbi n -cycles in X is given by $C_n^{orb}(X) = X^n /^{orb} \mathbb{Z}_n$.

In analogy with the example above we define the set of orbi-necklaces without a clasp with p beads and r colors to be $C_p^{orb}[r] = [r]^p /^{orb} \mathbb{Z}_p$. Its cardinality is given by

$$|C_p^{orb}[r]| = P_{\mathbb{Z}_p}^{orb}(r, r, \dots) = \frac{1}{p} \sum_{d|p} J_2(d) r^{\frac{p}{d}} = rp + \frac{r^p - r}{p}.$$

Next couple of results count explicitly the number of orbicycles in orbiquotient sets.

Theorem 7. If G acts on (X, f) then

$$|C_n^{orb}(X /^{orb} G)|_f = \frac{1}{n} \sum_{\alpha: [\frac{n}{d}] \rightarrow P(G)} \frac{J_2(d)}{|G|^{\frac{n}{d}}} \prod_{i=1}^{\frac{n}{d}} |\pi_{C(G)}(\alpha(i))| |X^{\alpha(i)}|_{fd}.$$

Proof.

$$\begin{aligned}
|C_n^{orb}(X/^{orb}G)|_f &= P_{\mathbb{Z}_n}^{orb}(|X/^{orb}G|_f, |X/^{orb}G|_{f^2}, \dots) \\
&= \frac{1}{n} \sum_{d|n} J_2(d) \left(|X/^{orb}G|_{fd}\right)^{\frac{n}{d}} \\
&= \frac{1}{n} \sum_{d|n} J_2(d) \left(\frac{1}{|G|} \sum_{(\bar{g}, h) \in P(G)} |\bar{g}| |X^{(g, h)}|_f\right)^{\frac{n}{d}} \\
&= \frac{1}{n} \sum_{d|n} \frac{J_2(d)}{|G|^{\frac{n}{d}}} \sum_{\alpha: [\frac{n}{d}] \rightarrow P(G)} \prod_{i=1}^{\frac{n}{d}} |\pi_{C(G)}(\alpha(i))| |X^{\alpha(i)}|_{fd} \\
&= \frac{1}{n} \sum_{\alpha: [\frac{n}{d}] \rightarrow P(G)} \frac{J_2(d)}{|G|^{\frac{n}{d}}} \prod_{i=1}^{\frac{n}{d}} |\pi_{C(G)}(\alpha(i))| |X^{\alpha(i)}|_{fd}.
\end{aligned}$$

□

Theorem 8. Let (X, f) be an \mathbb{A} -weighted set and $G \subseteq S_m$, then we have

$$|C_n^{orb}(X^{[m]}/^{orb}G)|_f = \frac{1}{n} \sum_{\alpha: [\frac{n}{d}] \rightarrow P(G)} \frac{J_2(d)}{|G|^{\frac{n}{d}}} \prod_{i=1}^{\frac{n}{d}} |X|_{fd}^{b(\alpha(i))}$$

where $|X|_{fd}^{b(\alpha(i))} = \prod_{k \geq 1} |X|_{fdk}^{b_k(C(\pi_1(\alpha(i))) \vee C(\pi_2(\alpha(i))))}$

Proof.

$$\begin{aligned}
|C_n^{orb}(X^{[m]}/^{orb}G)|_f &= P_{\mathbb{Z}_n}^{orb}(|X^{[m]}/^{orb}G|_f, |X^{[m]}/^{orb}G|_{f^2}, \dots) \\
&= \frac{1}{n} \sum_{d|n} J_2(d) P_G^{orb}(|X|_{fd}, |X|_{fd^2}, \dots)^{\frac{n}{d}} \\
&= \frac{1}{n} \sum_{d|n} J_2(d) \left(\frac{1}{|G|} \sum_{(\bar{g}, h) \in P(G)} \prod_{k \geq 1} |X|_{fdk}^{b_k(C(\bar{g}) \vee C(h))}\right)^{\frac{n}{d}} \\
&= \frac{1}{n} \sum_{d|n} \frac{J_2(d)}{|G|^{\frac{n}{d}}} \sum_{\alpha: [\frac{n}{d}] \rightarrow P(G)} \prod_{i=1, k \geq 1} |X|_{fdk}^{b_k(C(\pi_1(\alpha(i))) \vee C(\pi_2(\alpha(i))))} \\
&= \frac{1}{n} \sum_{\alpha: [\frac{n}{d}] \rightarrow P(G)} \frac{J_2(d)}{|G|^{\frac{n}{d}}} \prod_{i=1}^{\frac{n}{d}} |X|_{fd}^{b(\alpha(i))}.
\end{aligned}$$

□

Using similar methods one can count cycles on orbiquotient sets:

$$|C_n(X/G)|_f = \frac{1}{n} \sum_{\alpha: [\frac{n}{d}] \rightarrow G} \frac{J_2(d)}{|G|^{\frac{n}{d}}} \prod_{i=1}^{\frac{n}{d}} |X^{\alpha(i)}|_{f^d},$$

and

$$|C_n(X^m/G)|_f = \frac{1}{n} \sum_{\alpha: [\frac{n}{d}] \rightarrow G} \frac{J_2(d)}{|G|^{\frac{n}{d}}} \prod_{i=1}^{\frac{n}{d}} |X|_{f^d}^{c(\alpha(i))},$$

where $|X|_{f^d}^{c(g)} = \prod_{k \geq 1} |X|_{f^{dk}}^{c_k(g)}$ for $g \in G$.

5. Orbicycle index polynomial of S_n

A partition of depth k , denoted by $\alpha \vdash_k n$, of $n \in \mathbb{N}_+$ is a map $\alpha : (\mathbb{N}_+)^k \rightarrow \mathbb{N}$ such that

$$\sum_{(i_1, \dots, i_k) \in \mathbb{N}_+^k} i_1 \dots i_k \alpha(i_1, \dots, i_k) = n.$$

A partition of depth 1 is a partition in the usual sense. To each partition α we associate a canonical permutation of $[n]$ whose cycle structure is determined by α . Keeping this correspondence in mind one can check that if $\alpha \vdash n$ then

1. $Z(\alpha)$ is isomorphic to $\prod_{i=1}^n \mathbb{Z}_i^{\alpha_i} \times S_{\alpha_i}$.
2. If $h \in Z(\alpha)$, then $b_k(C(\alpha) \vee C(h)) = \sum_{d|k} c_k^d(\pi_d(h))$ where $\pi_d(h)$ is the projection of h into $\prod_{i=1}^n S_{\alpha_i}$.

Theorem 9.

$$P_{S_n}^{orb}(y_1, y_2, \dots, y_n) = \sum_{\beta \vdash_{-2} n} \prod_{(i,j,k) \in [n] \times [\alpha_i] \times [n]} \frac{y_k^{\sum_{d|k} \beta(d, \frac{k}{d})}}{j^{\beta(i,j)} \beta(i,j)!}.$$

Proof. By the previous remarks we have

$$\begin{aligned}
 P_{S_n}^{orb}(y_1, y_2, \dots, y_n) &= \frac{1}{n!} \sum_{(\bar{g}, h) \in P(S_n)} |\bar{g}| \prod_{k=1}^n y_k^{b_k(C(g) \vee C(h))} \\
 &= \frac{1}{n!} \sum_{\alpha \vdash n} \frac{n!}{\prod_{i=1}^n i^{\alpha_i} \alpha_i!} \sum_{h \in Z(\alpha)} \prod_{k=1}^n y_k^{b_k(C(\alpha) \vee C(h))} \\
 &= \sum_{\alpha \vdash n} \frac{1}{\prod_{i=1}^n \alpha_i!} \sum_{h \in \prod_{i=1}^n S_{\alpha_i}} \prod_{k=1}^n y_k^{\sum_{d|k} c_{\frac{k}{d}}(\pi_d(h))} \\
 &= \sum_{\alpha \vdash n} \frac{1}{\prod_{i=1}^n \alpha_i!} \sum_{\beta_i \vdash \alpha_i} \frac{\prod_{i=1}^n \alpha_i!}{\prod_{(i,j) \in [n] \times [\alpha_i]} j^{\beta(i,j)} \beta(i,j)!} \prod_{k=1}^n y_k^{\sum_{d|k} \beta_{\frac{k}{d}}(d)} \\
 &= \sum_{\beta \vdash 2n} \prod_{(i,j,k) \in [n] \times [\alpha_i] \times [n]} \frac{y_k^{\sum_{d|k} \beta(d, \frac{k}{d})}}{j^{\beta(i,j)} \beta(i,j)!}.
 \end{aligned}$$

Above we used the fact that $\sum_{d|k} c_{\frac{k}{d}}(\pi_d(h))$ depends only on the cycle structure of $\pi_d(h)$. □

The orbifold index polynomial can be use to compute the even dimensions of the orbifold cohomology groups for global orbifolds of the form $M^n /^{orb} G$, where M is a compact smooth manifold, and $G \subset S_n$. For simplicity we only consider cohomology in even dimensions. The orbifold cohomology is defined as follows:

$$H^{orb}(M^n / G) = \bigoplus_{\bar{g} \in C(G)} H((M^n)^g)^{Z(g)}.$$

The following result is a direct consequence of the characterization of the centralizer of permutations previously discussed.

Lemma 10.

$$H^{orb}(M^n / G) = \bigoplus_{\bar{g} \in C(G)} \bigotimes_i \left(H(M)^{\otimes_{c_i(g)}} \right)^{S_{c_i(g)}}.$$

Proof.

$$\begin{aligned}
 H^{orb}(M^n / G) &= \bigoplus_{\bar{g} \in C(G)} H((M^n)^g)^{Z(g)} \\
 &= \bigoplus_{\bar{g} \in C(G)} H\left(\prod_i M^{c_i(g)}\right)^{Z(g)} \\
 &= \bigoplus_{\bar{g} \in C(G)} \bigotimes_i \left(H(M)^{\otimes_{c_i(g)}} \right)^{S_{c_i(g)}}.
 \end{aligned}$$

□

Assume that we are given a finite basis X for $H(M)$, then we have the following result.

Theorem 11.

$$\dim \left(H^{orb} (M^n / S_n) \right) = P_G^{orb}(|X|, \dots, |X|).$$

Proof.

$$\begin{aligned} \dim \left(H^{orb} (M^n / S_n) \right) &= \sum_{\bar{g} \in C(G)} \prod_i \left| X^{c_i(g)} / S_{c_i(g)} \right| \\ &= \left| X^n /^{orb} S_n \right| \\ &= P_G^{orb}(|X|, \dots, |X|). \end{aligned}$$

□

Notice that above we use the trivial weight on X ; using a generic weight we obtain further information on the orbifold cohomology groups. Theorem 12 gives a combinatorial interpretation for the orbifold cohomology groups, however we do not have a combinatorial interpretation for the orbifold product introduced by Chen and Ruan in [9]. Until recently this problem seemed hopeless, however the alternative description of the Chen-Ruan product introduced by Jarvis, Kauffman and Kimura in [15] could pave the way for such a combinatorial understanding. Theorem 11 suggests the possibility of constructing, along the lines of [11], an orbi-analogue for the symmetric functions. This issue deserves further research.

6. Orbicycle index polynomial of D_n

The generators ρ and τ of the dihedral group $D_n = \{e, \rho, \dots, \rho^{n-1}, \tau, \dots, \tau\rho^{n-1}\}$ are such that $\rho^n = e$, $\tau^2 = e$ and $\tau\rho = \rho^{n-1}\tau$. The conjugacy classes of the dihedral groups are described in the following tables, see [10]. For n odd there are $\frac{n+3}{2}$ conjugacy classes organized in three families

Conjugacy class	Representative	Centralizer subgroup
$\{e\}$	e	D_n
$\{\rho^i, \rho^{-i}\}$ for $1 \leq i \leq \frac{n-1}{2}$	ρ^i	$\{\rho^i \mid 0 \leq i < n\}$
$\{\rho^i \tau \mid 0 \leq i < n\}$	τ	$\{e, \tau\}$

For n even there are $\frac{n+6}{2}$ conjugacy classes organized in five families

Conjugacy class	Representative	Centralizer subgroup
$\{e\}$	e	D_n
$\{\rho^{\frac{n}{2}}\}$	$\rho^{\frac{n}{2}}$	D_n
$\{\rho^i, \rho^{-i}\}, 1 \leq i < \frac{n}{2}$	ρ^i	$\{\rho^i \mid 0 \leq i < n\}$
$\{\rho^{2i} \tau \mid 0 \leq i < \frac{n}{2}\}$	τ	$\{e, \tau, \rho^{\frac{n}{2}}, \rho^{\frac{n}{2}} \tau\}$
$\{\rho^{2i+1} \tau \mid 0 \leq i < \frac{n}{2}\}$	$\rho \tau$	$\{e, \rho \tau, \rho^{\frac{n}{2}}, \rho^{\frac{n}{2}+1} \tau\}$

For a_1, \dots, a_k maps from a subset of \mathbb{N}^+ into \mathbb{N}^+ , and n in this subset we set

$$\varphi(a_1(n), \dots, a_k(n), n) = |\{x \in \mathbb{Z}_{a_1} \times \dots \times \mathbb{Z}_{a_k} \mid (x, n) = d\}|.$$

Proceeding as in the previous sections one can show that

$$\varphi(a_1(n), \dots, a_k(n), n) = \sum_{d|n} a_1(d) \dots a_k(d) \mu\left(\frac{n}{d}\right),$$

also for $f : \{d : d \mid n\} \rightarrow \mathbb{A}$ an arbitrary map we have that

$$\sum_{x \in \mathbb{Z}_{a_1} \times \dots \times \mathbb{Z}_{a_k}} f((x, n)) = \sum_{d|n} \varphi(a_1(n), \dots, a_k(n), n) f(d).$$

We will use two instances of the identity above, namely, let $n \in \mathbb{N}^+$ be odd, $a_1(n) = \frac{n-1}{2}$, and $a_2(n) = n-1$, then

$$\varphi(a_1(n), a_2(n), n) = \sum_{d|n} \left(\frac{d^2 - 2d + 1}{2}\right) \mu\left(\frac{n}{d}\right) = \frac{J_2(n) - 2\varphi(n)}{2}.$$

Let $n \in \mathbb{N}^+$ be even, $a_1(n) = \frac{n-2}{2}$, and $a_2(n) = n-1$, then we get

$$\varphi(a_1(n), a_2(n), n) = \frac{J_2(n) - 3\varphi(n)}{2}.$$

Theorem 12. Let $n \in \mathbb{N}_+$ be odd. The orbicycle index polynomial of D_n is given by

$$P_{D_n}^{orb} = \frac{1}{2} P_{\mathbb{Z}_n}^{orb} - \frac{3}{2} P_{\mathbb{Z}_n} + \frac{3}{2} x_1 x_2^{\frac{n-1}{2}}.$$

Proof.

$$\begin{aligned}
 P_{D_n}^{orb}(x_1, \dots, x_n) &= \frac{1}{2n} \sum_{(\bar{g}, h) \in P(D_n)} |\bar{g}| \prod_{k=1}^n x_k^{b_k(C(\bar{g}) \vee C(h))} \\
 &= P_{D_n} + \frac{1}{n} \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=0}^{n-1} \prod_{k=1}^n x_k^{b_k(C(\rho^i) \vee C(\rho^j))} + x_1 x_2^{\frac{n-1}{2}} \\
 &= P_{D_n} + \frac{1}{n} \sum_{i=0}^{\frac{n-1}{2}} \sum_{j=0}^{n-1} \prod_{k=1}^n x_k^{b_k(C(\rho^i) \vee C(\rho^j))} - P_{\mathbb{Z}_n} + x_1 x_2^{\frac{n-1}{2}} \\
 &= \frac{1}{2} P_{\mathbb{Z}_n} + \frac{3}{2} x_1 x_2^{\frac{n-1}{2}} + \frac{1}{n} \sum_{i=0}^{\frac{n-1}{2}} \sum_{j=0}^{n-1} \prod_{k=1}^n x_k^{b_k(C(\rho^i) \vee C(\rho^j))} - P_{\mathbb{Z}_n} \\
 &= \frac{1}{2n} \sum_{d|n} (J_2(d) - 2\varphi(d)) x_d^{\frac{n}{d}} - \frac{1}{2} P_{\mathbb{Z}_n} + \frac{3}{2} x_1 x_2^{\frac{n-1}{2}} \\
 &= \frac{1}{2} P_{\mathbb{Z}_n}^{orb} - \frac{3}{2} P_{\mathbb{Z}_n} + \frac{3}{2} x_1 x_2^{\frac{n-1}{2}}.
 \end{aligned}$$

□

Recall that τ and ρ are given $\tau(x) = 3 - x$ and $\rho^r(x) = x + r$. Our next table gives the equivalence class of $x \in \mathbb{Z}_n$ under five different equivalence relations.

Partition	Equivalence class
$C(\tau) \vee C(\rho^{\frac{n}{2}})$	$\left\{ x, 3 - x, x + \frac{n}{2}, 3 - x + \frac{n}{2} \right\}$
$C(\tau) \vee C(\rho^{\frac{n}{2}} \tau)$	$\left\{ x, 3 - x, 3 - x + \frac{n}{2}, x + \frac{n}{2} \right\}$
$C(\rho\tau) \vee C(\rho^{\frac{n}{2}})$	$\left\{ x, 4 - x, x + \frac{n}{2}, 4 - x + \frac{n}{2} \right\}$
$C(\rho\tau) \vee C(\rho^{\frac{n}{2}+1}\tau)$	$\left\{ x, 4 - x, 4 - x + \frac{n}{2}, x + \frac{n}{2} \right\}$
$C(\rho^{\frac{n}{2}}) \vee C(\tau\rho^i)$	$\left\{ x, x + \frac{n}{2}, 3 - i - x, 3 - i - x + \frac{n}{2} \right\}$

So we see that the equivalence class of $x \in [n]$ under the equivalence relation $C(\rho^{\frac{n}{2}}) \vee C(\tau\rho^i)$ is

$$\bar{x} = \left\{ x, x + \frac{n}{2}, 3 - i - x, 3 - i - x + \frac{n}{2} \right\},$$

so that $|\bar{x}| \in \{2, 4\}$. It is not difficult to see that $|\bar{x}| = 2$ if and only if either $2x \equiv 3 - i$ or $2x \equiv 3 - i + \frac{n}{2}$. Therefore $b_2(C(\rho^{\frac{n}{2}}) \vee C(\tau\rho^i))$ is 1 if $\frac{n}{2}$ is odd, 2 if $\frac{n}{2}$ is even and i is odd, and 0 if $\frac{n}{2}$ is even and i is even.

Theorem 13. Let $n \in \mathbb{N}_+$ be even. According to whether n is 0 or 2 mod 4, the orbifold index polynomial of D_n is given by

$$P_{D_n}^{orb} = \frac{1}{2}P_{\mathbb{Z}_n}^{orb} - 2P_{\mathbb{Z}_n} + \frac{3n+4}{8n} \left(x_2^{\frac{n}{2}} + x_1^2 x_2^{\frac{n-2}{2}} \right) \\ + \frac{1}{2n} \sum_{d|\frac{n}{2}} \varphi(n-1, \frac{n}{2d}) x_{\frac{n}{d}}^d + \begin{cases} \frac{2n+2}{n} x_2 x_4^{\frac{n-2}{4}} \\ \frac{n+1}{n} x_4^{\frac{n}{4}} + \frac{1}{2} x_2^2 x_4^{\frac{n-4}{2}} + \frac{n+2}{2n} x_2^2 x_4^{\frac{n-4}{4}} \end{cases}$$

Proof.

$$P_{D_n}^{orb} = \frac{1}{2n} \sum_{(\bar{g}, h) \in P(D_n)} |\bar{g}| \prod_{k=1}^n x_k^{b_k(C(\bar{g}) \vee C(h))} \\ = P_{D_n} + \frac{1}{2n} \sum_{h \in D_n} \prod_{k=1}^n x_k^{b_k(C(\rho^{\frac{n}{2}}) \vee C(h))} \\ + \frac{1}{n} \sum_{i=0}^{\frac{n-2}{2}} \sum_{j=0}^{n-1} \prod_{k=1}^n x_k^{b_k(C(\rho^i) \vee C(\rho^j))} - P_{\mathbb{Z}_n} \\ + \frac{n+2}{4n} \sum_{h \in \{e, \tau, \rho^{\frac{n}{2}}, \rho^{\frac{n}{2}} \tau\}} \prod_{k=1}^n x_k^{b_k(C(\tau) \vee C(h))} \\ + \frac{n+2}{4n} \sum_{h \in \{e, \rho\tau, \rho^{\frac{n}{2}}, \rho^{\frac{n}{2}+1} \tau\}} \prod_{k=1}^n x_k^{b_k(C(\rho\tau) \vee C(h))}.$$

We compute the last four summands in the expression above

$$\sum_{h \in D_n} \prod_{k=1}^n x_k^{b_k(C(\rho^{\frac{n}{2}}) \vee C(h))} = \sum_{i=0}^{n-1} \prod_{k=1}^n x_k^{b_k(C(\rho^{\frac{n}{2}}) \vee C(\rho^i))} + \sum_{i=0}^{n-1} \prod_{k=1}^n x_k^{b_k(C(\rho^{\frac{n}{2}}) \vee C(\tau\rho^i))} \\ = \sum_{i=0}^{n-1} x_{\frac{n}{2}}^{\binom{n}{2}, i} + \sum_{i=0}^{n-1} \prod_{k=1}^n x_k^{b_k(C(\rho^{\frac{n}{2}}) \vee C(\tau\rho^i))} \\ = \sum_{d|\frac{n}{2}} \varphi(n-1, \frac{n}{2d}) x_{\frac{n}{d}}^d + \begin{cases} n x_2 x_4^{\frac{n-2}{4}} \\ \frac{n}{2} x_4^{\frac{n}{4}} + \frac{n}{2} x_2^2 x_4^{\frac{n-4}{4}} \end{cases}$$

$$\sum_{i=0}^{\frac{n-2}{2}} \sum_{j=0}^{n-1} \prod_{k=1}^n x_k^{b_k(C(\rho^i) \vee C(\rho^j))} = \sum_{i=0}^{\frac{n-2}{2}} \sum_{j=0}^{n-1} x_{\frac{n}{(i,j,n)}}^{(i,j,n)} = \sum_{d|n} \left(\frac{J_2(d) - 3\varphi(d)}{2} \right) x_d^{\frac{n}{d}} \\ = \frac{n}{2} P_{\mathbb{Z}_n}^{orb} - \frac{3n}{2} P_{\mathbb{Z}_n}.$$

$$\sum_{h \in \{e, \tau, \rho^{\frac{n}{2}}, \rho^{\frac{n}{2}}\tau\}} \prod_{k=1}^n x_k^{b_k(C(\tau) \vee C(h))} = 2x_2^{\frac{n}{2}} + 2 \begin{cases} x_2 x_4^{\frac{n-2}{4}} \\ x_4^{\frac{n}{4}} \end{cases}$$

$$\sum_{h \in \{e, \rho\tau, \rho^{\frac{n}{2}}, \rho^{\frac{n}{2}+1}\tau\}} \prod_{k=1}^n x_k^{b_k(C(\rho\tau) \vee C(h))} = 2x_1^2 x_2^{\frac{n-2}{2}} + 2 \begin{cases} x_2 x_4^{\frac{n-2}{4}} \\ x_2^2 x_4^{\frac{n}{4}-1} \end{cases}$$

□

In this work we have extended Pólya theory to the context of orbiquotient sets. The main ingredient of the new theory is the orbicycle index polynomial which we computed in various cases. We expect that our construction will find applications in the study of the topology of orbifolds and also in the theory of species. It would be interesting to search for a further extension of Pólya theory within the context of rational combinatorics introduced in [5, 6] based on the previous work [12] and further discussed in [7]. One should obtain a generalization of Pólya theory in which finite sets are replaced by finite groupoids [3].

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