Global Journal of Pure and Applied Mathematics
ISSN 0973-1768 Volume 5, Number 3 (2009), pp. 217–228

http://www.ripublication.com/gjpam.htm

The First Nonprincipal Eigenvalue for a Steklov Problem

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Abstract

In this paper we prove the existence of a first nonprincipal eigenvalue for the Steklov prblem $\Delta_p u = |u|^{p-2} u$ in $\Omega, |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda[m(u^+)^{p-1} - n(u^-)^{p-1}]$ on $\partial\Omega$. Also we give an other variation characterization for the second eigenvalue of problem $\Delta_p u = |u|^{p-2} u$ in $\Omega, |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda m |u|^{p-2} u$ on $\partial\Omega$.

AMS Subject Classification: 35J70, 35P30.

Keywords and phrases: Nonprincipal eigenvalue; Steklov problem; p-Laplacian; weights; mountain pass.

Introduction

In a previous work [1], we investigated the following asymmetric Steklov problem with weights:

$$\begin{cases} \triangle_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda[m(u^+)^{p-1} - n(u^-)^{p-1}] & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν denotes the unit exterior normal, $1 < p < \infty$ and $\Delta_p u = div(|\nabla u|^{p-2} \nabla u)$ indicate the $p-$Laplacian. $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a Lipschitz continuous boundary, where $N \geq 2, m, n \in L^q(\partial\Omega)$ with $\frac{N-1}{p-1} < q < \infty$ if $p < N$ and

$q \geq 1$ if $p \geq N$. We proved the existence of a first nonprincipal positive eigenvalue for (1.1) in case where the weights m and n have meanvalues nonzero. As an application we gave another variational characterization for the second eigenvalue of the problem (1.1) with $m = n$.

The construction of this distinguished eigenvalue was obtained by applying a version of the mountain pass theorem to the functional $\phi(u) := \frac{1}{p}\int_\Omega |\nabla u|^p \, dx$ restricted to the manifold $M_{m,n} := \{u \in W^{1,p}(\Omega); \frac{1}{p}\int_{\partial\Omega}[m(u^+)^p + n(u^-)^p] d\sigma = 1\}$. In this process the (PS) condition was show to hold at all levels and the geometry of the mountain pass was derived from the observation that $\varphi(m)$ and $-\varphi(n)$ were strict local minima (where $\varphi(m)$ denotes the normalized positive first eigenvalue of the problem (1.1) with $m = n$). In [2], we are interested in the singular case (in case where one of the weights has meanvalue zero). In this case the Palais Smale condition is not satisfied any more at level 0 and at least one of the two naturals candidates for local minimum fails to belong to the manifold $M_{m,n}$. To by pass this difficulty we applied a version of the mountain pass theorem for a local C^1 functional restricted to a C^1 manifold and which satisfies the Palais Smale condition of Cerami at certain levels.

Our purpose in the present paper is to study the following asymmetric Steklov problem:

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda[m(u^+)^{p-1} - n(u^-)^{p-1}] & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

When trying to adapt the approach followed in [1] to the present situation, this relevant functional is $A(u) = \frac{1}{p}\|u\|_{1,p}^p$ restricted to the manifold $M_{m,n}$ where $\|u\|_{1,p} = (\int_\Omega |\nabla u|^p \, dx + \int_\Omega |u|^p \, dx)^{1/p}$ is the $W^{1,p}(\Omega)$-norm. We show the existence of a first nonprincipal positive eigenvalue, as an application we give another variational characterization for the second eigenvalue for (1.2) with $m = n$.

Preliminaries

throughout this paper Ω will be bounded domain in $\mathbb{R}^{\mathbb{N}}$. We assume that $m, n \in L^q(\partial\Omega), m^+ = \max(m, 0) \neq 0, n^+ = \max(n, 0) \neq 0,$ where $q > \frac{N-1}{p-1}$ if $1 < p \leq N$ and $q \geq 1$ if $p > N$. We are interested in weak solution of (1.2) i.e., functions $u \in W^{1,p}(\Omega)$ satisfying

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx + \int_\Omega |u|^{p-2}u\varphi dx = \lambda \int_{\partial\Omega}[m(u^+)^{p-1} - n(u^-)^{p-1}]\varphi d\sigma \quad \forall \varphi \in W^{1,p}(\Omega). \tag{2.1}$$

Where $d\sigma$ is the $N-1$ dimensional Hausdorff measure. Let us formulate variationally the problem (1.2). For that purpose we introduce the C^1 functionals A and $B_{m,n}: W^{1,p}(\Omega) \to \mathbb{R}$, defined by $A(u) = \frac{1}{p} \| u \|_{1,p}^p$ and $B_{m,n}(u) = \frac{1}{p} \int_{\partial\Omega} [m(u^+)^p + n(u^-)^p] d\sigma$. At this point let us introduce the set $M_{m,n} := \{u \in W^{1,p}(\Omega); B_{m,n}(u) = 1\}$. The condition $m^+ \neq 0$ implies that $M_{m,n} \neq \emptyset$. Moreover the set $M_{m,n}$ is a C^1 manifold in $W^{1,p}(\Omega)$; for any $u \in M_{m,n}$ the tangent space of $M_{m,n}$ at $u, T_u M_{m,n}$ is the set $T_u M_{m,n} := \{w \in W^{1,p}(\Omega) : \left\langle B'_{m,n}(u), w \right\rangle = 0\}$. Let us denote by $\tilde{A}$ the restriction of A to $M_{m,n}$. We recall that a value c is a critical value of $\tilde{A}$ if $A'(u)/T_u M_{m,n} = 0$ and $\tilde{A}(u) = c$ for some $u \in M_{m,n}$. Let us briefly recall some properties relative to the following Steklov problem

$$\begin{cases} \triangle_p u = |u|^{p-2} u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda m |u|^{p-2} u & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Each eigenvalue of (2.2) is called a Steklov eigenvalue. In [3] Bonder and Rossi proved by using a general result from the infinitely dimensional Ljuternik-Schnirelman theory (see [5]) that for any $k \in \mathbb{N}^*, \lambda_k(m)$ defined by

$$\frac{1}{\lambda_k(m)} = \sup_{C \in C_k} \min_{u \in C} \frac{\| u \|_{L^p_m(\partial\Omega)}^P}{\| u \|_{1,p}^p}$$

is a nondecreasing and unbounded sequence of the positive eigenvalue of the problem (2.2), where $\| u \|_{L^p_m(\partial\Omega)}^p = \int_{\partial\Omega} m |u|^p \, dx, C_k = \{C \subset W^{1,p}(\Omega) : C$ compact, symmetric and $\gamma(C) \geq k\}$ with $\gamma(C)$ denotes the Krasnoselski's genus on $W^{1,p}(\Omega)$. He showed the following proposition concerning the first Steklov eigenvalue.

Proposition 2.1. The first eigenvalue $\lambda_1(m)$ of (2.2) is simple and isolated. Moreover any associated eigenfunction does not change sign in Ω.

We can easily show that $\lambda_k(m) = \inf_{C \in \Gamma_k} \sup_{u \in C} \| u \|_{1,p}^p$, where

$\Gamma_k = \{C \subset M_{m,n} : C$ compact, symmetric and $\gamma(C) \geq k\}$.

Let E be a Banach space and let $M = \{u \in E; g(u) = 1\}$, where $g \in C^1(E, \mathbb{R})$ and 1 is a regular value of g. Let $f \in C^1(E, \mathbb{R})$ and consider the restriction $\tilde{f}$ to f. The differential of $\tilde{f}$ at $u \in M$ has a norm which will be denoted by $\| \tilde{f} \|_*$ and which is given by the norm of the restriction $\tilde{f}$ to the tangent space $T_u(M) := \{v \in E; \left\langle g'(u), v \right\rangle = 0\}$; where $\langle , \rangle$ denotes the duality pairing between E^*

and E. We recall that $\tilde{f}$ is said to satisfy the Palais-Smale condition on M if, for any sequences $u_k \in M$ such that $\tilde{f}(u_k)$ is bounded and $\| \tilde{f}'(u_k) \|_* \to 0$, one has that u_k admits a converging subsequences. The following Lemma will be used in the proof of our theorem. It guarantees the existence of a critical point in any component of any sublevel set.

Lemma 2.2 ([4]): Let E, g, M, f and $\tilde{f}$ be as considered previously. Assume $\tilde{f}$ is bounded from below on M and satisfy the Palais-Smale condition on M. Let $d \in \mathbb{R}$ and consider $O = \{u \in M; \tilde{f}(u) < d\}$. Then any component (i.e. a nonempty maximal open connected subset) O_1 of O contains a critical point of $\tilde{f}$.

We now recall a version of the classical mountain pass theorem on a C^1 manifold.

Proposition 2.3 ([4]): Let $u, v \in M$ with $u \neq v$ and suppose that

$$H = \{h \in C([0,1], M); h(0) = u \text{ and } h(1) = v\}$$

is nonempty. Assume that

$$c := \inf_{h \in H} \max_{w \in h([0,1])} f(w) > \max(f(u), f(v)),$$

and that $\tilde{f}$ satisfies the Palais-Smale condition on M. Then c is a critical point of $\tilde{f}$.

We will apply Proposition (2.3) with $E = W^{1,p}(\Omega), M = M_{m,n}, f = A$ and $g = B_{m,n}$.

A first nonprincipal eigenvalue

Now consider the family of paths in $M_{m,n}$:

$$\Gamma = \{\gamma \in C([0,1], M_{m,n}) : \gamma(0) \leq 0 \text{ and } \gamma(1) \geq 0\},$$

and define the minimax value

$$c(m,n) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[0,1]} \tilde{A}(u), \tag{3.1}$$

Since $\Gamma \neq \emptyset$ (see [1]), then $c(m,n)$ is finite.

Theorem 3.1. $c(m,n)$ is an eigenvalue of (1.2) which satisfies

$$\max\{\lambda_1(m), \lambda_1(n)\} < c(m,n)..$$

Moreover there is no eigenvalue of (1.2) between $\max\{\lambda_1(m), \lambda_1(n)\}$ and $c(m,n)$.

As the demonstration is comparatively long, we organize it in several propositions and lemmas.

Proposition 3.2. $\tilde{A}$ satisfies the Palais-Smale condition (PS) on $M_{m,n}$.

Proof: Let $(u_k) \in M_{m,n}$ be a sequence such that $\tilde{A}(u_k) \to c$, where c is a positive constant and

$$| \int_\Omega | \nabla u_k |^{p-2} \nabla u_k \nabla \varphi dx + \int_\Omega | u_k |^{p-2} u_k \varphi dx | \le \varepsilon_k \| \varphi \|_{1,p} \quad \forall \varphi \in T_{u_k} M_{m,n}, \tag{3.2}$$

with $\lim_{k\to+\infty} \varepsilon_k = 0$. Since the sequence (u_k) is bounded then there exists a subsequence still denoted by (u_k) such that $u_k \to u_0$ weakly in $W^{1,p}(\Omega)$ and $u_k \to u_0$ strongly in $L^p(\Omega)$. Let us write for $w \in W^{1,p}(\Omega)$

$$a_k(w) = w - \left[\int_{d\Omega} (m(u_k^+)^{p-1} - n(u_k^-)^{p-1}) w \right] u_k \in T_{u_k} M_{m,n}.$$

Taking $\varphi = a_k(w)$ in (3.2), one deduces

$$\left| \int_\Omega |\nabla u_k|^{p-2} \nabla u_k \nabla w dx + \int_\Omega |u_k|^{p-2} u_k w dx - \left[\int_{\partial\Omega} (m(u_k^+)^{p-1} - n(u_k^-)^{p-1}) w \right] \int_\Omega |\nabla u_k|^p \right|$$
$$\le \varepsilon_k ||w|| (1 + C||u_k||^p), \tag{3.3}$$

for some constant C. Indeed: $\| a_k(w) \| = \| w - c_k u_k \|$, where $c_k = \int_{\partial\Omega} (m(u_k^+)^{p-1} - n(u_k^-)^{p-1}) w d\sigma$, so $\| a_k(w) \| \le \| w \| + | c_k | . \| u_k \|$ and $| c_k | \le \int_{\partial\Omega} m(u_k^+)^{p-1} w d\sigma | + | \int_{\partial\Omega} n(u_k^-)^{p-1} w d\sigma |$. Using the compacity of the trace mapping $W^{1,p}(\Omega) \to L^{\frac{pq}{q-1}}(\partial\Omega)$, we obtain

$$| \int_{\partial\Omega} m(u_k^+)^{p-1} w d\sigma | \le c_1(m). \| w \| . \| u_k^+ \|^{p-1},$$

where $c_1(m)$ is a constant depending of m. Similarly we have

$$| \int_{\partial\Omega} n(u_k^-)^{p-1} w d\sigma | \le c_1(n). \| w \| . \| u_k^- \|^{p-1} .$$

Consequently $| c_k | \le C \| w \| . \| u_k \|^{p-1}$, where $C \ge \max\{c_1(m), c_1(n)\}$. Thus

$$\| a_k(w) \| \le \| w \| (1 + C \| u_k \|^p).$$

Finally, we have (3.3). Put $w = u_k = u_0$ in inequality (3.3), one obtains

$$\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla(u_k-u_0)dx+\int_\Omega |u_k|^{p-2}u_k(u_k-u_0)dx \to 0 \text{ as } k\to +\infty.$$

It then follows from the (S^+) property that $u_k \to u_0$ strongly in $W^{1,p}(\Omega)$.

Lemma 3.3. Let $v_k \in W^{1,p}(\Omega)$ with $v_k \geq 0, v_k \neq 0$ and $|v_k > 0| \to 0$. Let n_k be a bounded in $L^q(\partial\Omega)$. Then $\int_{\partial\Omega} n_k v_k^p d\sigma / \| v_k \|_{1,p}^p \to 0.$.

Proof: Without loss of generality, one can assume that $\| v_k \|_{1,p} = 1$. So for a subsequence, $v_k \to v$ weakly in $W^{1,p}(\Omega)$ strongly in $L^p(\Omega)$. In addition $v_k \to v$ strongly in $L^{\frac{pq}{q-1}}(\partial\Omega)$ and $v_k(x) \to v(x)$ a.e. in Ω. The assumption $|v_k > 0| \to 0$ implies $v(x) = 0$ a.e. $x \in \Omega$ since $v_k(x) \to 0$ at measure. Consequently $|\int_{\partial\Omega} n_k v_k^p d\sigma| \leq \| n_k \| L_{q(\partial\Omega)} \| v_k \|^p_{L^{\frac{pq}{q-1}(\partial\Omega)}} \to 0$ (since $v_k \to 0$ strongly in $L^{\frac{pq}{q-1}}(\partial\Omega)$).

We now turn to the geometry of $\tilde{A}$. Let φ_m be the normalized positive eigenfunction associated to the first positive eigenvalue $\lambda_1(m)$.

Proposition 3.4. If $m \in L^q(\partial\Omega)$ and $m^+ \neq 0$, then $\varphi_m \in M_{m,n}$ is a strict local minimum of $\tilde{A}$, with in addition for some $\varepsilon_0 > 0$ and all $0 < \varepsilon < \varepsilon_0$,

$$\tilde{A}(\varphi_m) = \lambda_1(m) < \inf\{\tilde{A}(u); u \in M_{m,n} \cap \partial B(\varphi_m, \varepsilon)\} \tag{3.4}$$

where $B(\varphi_m, \varepsilon)$ denotes the ball in $W^{1,p}(\Omega)$ of center φ_m and radius ε. Similar conclusion for $-\varphi_n$ if $n^+ \neq 0$.

Proof: One first shows for some $\varepsilon_0 > 0$, ,

$$\tilde{A}(\varphi_m) < \tilde{A}(u) \quad \forall u \in M_{m,n} \cap B(\varphi_m, \varepsilon_0\}, u \neq \varphi_m. \tag{3.5}$$

Assume by contradiction the existence of a sequence $u_k \in M_{m,n}$ with $u_k \neq \varphi_m, u_k \to \varphi_m$ strongly in $W^{1,p}(\Omega)$ and $\tilde{A}(u_k) \leq \lambda_1(m)$. We first observe that u_k changes sign for k sufficiently large. Indeed, since $u_k \to \varphi_m, u_k$ must be > 0 somewhere. If $u_k \geq 0$ in Ω, then

$$\tilde{A}(u_k) = \frac{1}{p} \| u_k \|_{1,p}^p > \frac{\lambda_1(m)}{p} \int_{\partial\Omega} m |u_k|^p \, d\sigma = \lambda_1(m)$$

since $u_k \neq \varphi_m$, but this conradicts $\tilde{A}(u_k) \leq \lambda_1(m)$. So u_k changes sign for k sufficiently large. Now we have

$$\begin{aligned} \frac{\lambda_1(m)}{p} \int_{\partial\Omega} [m(u_k^+)^p + n(u_k^-)^p] d\sigma &= \lambda_1(m) \\ &\geq \tilde{A}(u_k) \\ &= \frac{1}{p} \| u_k^+ \|_{1,p}^p + \frac{1}{p} \| u_k^- \|_{1,p}^p \\ &\geq \frac{\lambda_1(m)}{p} \int_{\partial\Omega} m(u_k^+)^p d\sigma + \frac{1}{p} \| u_k^- \|_{1,p}^p. \end{aligned} \tag{3.6}$$

Thus (3.6) implies

$$\frac{\int_{\partial\Omega} n(u_k^-)^p \, d\sigma}{\| u_k^- \|_{1,p}^p} \geq \frac{1}{\lambda_1(m)}.$$

Since $u_k \to \varphi_m, | u_k^- > 0 | \to 0$. By lemma 3.3, we have $\frac{\int_{\partial\Omega} n(u_k^-)^p \, d\sigma}{\| u_k^- \|_{1,p}^p} \to 0$. It follows that $0 \geq \frac{1}{\lambda\,(m)}$, this is a contradiction. Thus the proposition (3.2) is proved. Now we show (3.4), one assumes by contradiction the existence of $0 < \varepsilon < \varepsilon_0$ and of a sequence $u_k \in M_{m,n}$ such that $\| u_k - \varphi_m \|_{1,p} = \varepsilon$ and $\tilde{A}(u_k) \leq \tilde{A}(\varphi_m) + \frac{1}{2k^2}$. Consider

$$C := \{ u \in M_{m,n}; \varepsilon - \delta \leq \| u_k - \varphi_m \| \leq \varepsilon + \delta \},$$

where $0 < \delta < \varepsilon$ and $\delta + \varepsilon < \varepsilon_0$. Clearly $\inf\{\tilde{A}(u); u \in C\} = \tilde{A}(\varphi_m)$. We apply for each k Ekland's variational principle to the functional $\tilde{A}$ on C to get the existence of a sequence $v_k \in C$ such that

$$\tilde{A}(v_k) \leq \tilde{A}(u_k) \leq \tilde{A}(\varphi_m) + \frac{1}{2k^2} \tag{3.7}$$

$$\| v_k - u_k))_{1,p} \leq \frac{1}{k} \tag{3.8}$$

$$\tilde{A}(v_k) \leq \tilde{A}(u_k) + \frac{1}{k} \| u - v_k \|_{1,p} \qquad \forall u \in C. \tag{3.9}$$

Our purpose is to show that v_k is a Palais-Smale sequence for $\tilde{A}$, i.e. that $\tilde{A}(v_k)$ is bounded (which is clearly by (3.7)) and that $\|\tilde{A}'(v_k)\|_* \to 0$. We fix k with $\frac{1}{k} < \delta$, take $\omega \in T_{v_k}, M_{m,n}$ and consider a C^1 path $\gamma :]-\eta, \eta[\to M_{m,n}$ such that $\gamma(0) = v_k$ and $y'(0) = \omega$. For $|t|$ sufficiently small, $\gamma(t) \in C$. Indeed

$$\lim_{t\to\infty} \|\gamma(t) - \varphi_m\|_{1,p} = \|v_k - \varphi_m\|_{1,p}, \tag{3.10}$$

and it is easily seen using (3.8), $0 < \frac{1}{k} < \delta$ and $|u_k - \varphi_m\|_{1,p} = \varepsilon$ that the right-hand side of (3.10) is $> \varepsilon - \delta$ and $< \varepsilon + \delta$. So we can take $u = \gamma(t)$ in (3.9). This gives, for $t > 0$,

$$\frac{\tilde{A}(v_k) - \tilde{A}(\gamma(t))}{t} \le \frac{1}{k} \left\| \frac{\gamma(t) - v_k}{t} \right\|,$$

and so, going to the limit as $t \to 0$, we get

$$-\left\langle \tilde{A}'(v_k), \omega \right\rangle \le \frac{1}{k} \|\omega\|.$$

Consequently $\|\tilde{A}'(v_k)\|_* \le \frac{1}{k}$. Thus $\|\tilde{A}'(v_k)\|_* \to 0$ and v_k is Palais-Smale sequence for $\tilde{A}$. It follows that, for a subsequence, $v_k \to v$ strongly in $W^{1,p}(\Omega)$. Clearly $v \in C$ and satisfies $\|v - \varphi_m\|_{1,p} = \varepsilon$ and $\tilde{A}(v) = \tilde{A}(\varphi_m)$ which contradicts (3.5), similar argument when $n^+ \neq 0$.

Lemma 3.5. $c(m,n) > \max\{\lambda_1(m), \lambda_1(n)\}$.

Proof: One first shows that $\lambda_1(m) \le c(m,n)$ and $\lambda_1(n) \le c(m,n)$. For any $\gamma \in \Gamma, \gamma(1) \in M_{m,n}$ is ≥ 0 and so satisfies the contraint in

$$\lambda_1(m) = \inf\left\{ \frac{1}{p} \|u\|_{1,p}^p ; u \in W^{1,p}(\Omega) \ \ and \ \ \frac{1}{p} \int_{\partial\Omega} m|u|^p \, d\sigma = 1 \right\}.$$

Consequently $\lambda_1(m) \le c(m,n)$, and a similar argument applies to $\lambda_1(n)$. Now one shows the strict inequality $\lambda_1(m) < c(m,n)$. Assume by contradiction that $\lambda_1(m) = c(m,n)$, so there exists a sequence $\gamma_k \in \Gamma$ such that $\max_{t\in[0,1]} \tilde{A}(\gamma_k(t)) \to \lambda_1(m)$. Put $u_k = \gamma_k(1)$, since $u_k \ge 0$ one has

$$\lambda_1(m) \le \frac{1}{p} \|u_k\|_{1,p}^p \le \max_{t\in[0,1]} \tilde{A}(\gamma_k(t)) \to \lambda_1(m), \tag{3.11}$$

and consequently $\frac{1}{p}\| u_k \|_{1,p}^p \to \lambda_1(m)$. Thus $\| u_k \|_{1,p}$ remains bounded, for a subsequence and for some $u_0 \in W^{1,p}(\Omega)$, one has $u_k \to u_0$ weakly in $W^{1,p}(\Omega)$. Since $u_k \geq 0$ and using the trace mapping, one has $\frac{1}{p}\int_{d\Omega} m |u_0|^p \, d\sigma = 1$. Thus $u_0 \in M_{m,n}$ and so $\lambda_1(m) \leq \frac{1}{p}\| u_0 \|_{1,p}^p \leq \liminf \frac{1}{p}\| u_k \|_{1,p}^p = \lambda_1(m)$, which implies that $\frac{1}{p}\| u_0 \|_{1,p}^p = \lambda_1(m)$. Consequently $u_k \to u_0$ strongly in $W^{1,p}(\Omega)$ and we conclude that $u_0 = \varphi_m$. Let us now choose $\varepsilon > 0$ such that (3.5) holds and $B(\varphi_m, \varepsilon)$ does not contain any function v with $v \leq 0$, which is clearly possible. For k sufficiently large $u_k = \gamma_k(1) \in B(\varphi_m, \varepsilon)$, while $\gamma_k(0) \notin B(\varphi_m, \varepsilon)$, since $\gamma_k(0) \leq 0$. It follows that the path γ_k intersects $\partial B(\varphi_m, \varepsilon)$ and consequently $\max_{t\in[0,1]} \tilde{A}(\gamma_k(t)) \geq \inf\{\tilde{A}(u); u \in M_{m,n} \cap \partial B(\varphi_m, \varepsilon\} > \lambda_1(m)$. This contradicts (3.11).

Lemma 3.6. If $m_k \in L^q(\partial\Omega)$ with $m_k^+ \neq 0$ and $m_k^+ \to 0$ in $L^q(\partial\Omega)$, then $\lambda_1(m_k) \to +\infty$.

Proof: By definition of $\lambda_1(m_k)$ and the trace mapping we have

$$\frac{1}{\lambda_1(m_k)} = \frac{\int_{\partial\Omega} m_k |\varphi_{m_k}|^p \, d\sigma}{\| \varphi_{m_k} \|_{1,p}^p} \leq C\left(\int_{\partial\Omega} (m_k^+)^q\right)^{\frac{1}{q}}.$$

Lemma 3.7. For any $d > 0$, the set

$$O := \left\{ u \in M_{m,n}; u \geq 0 \ \ et \ \ \tilde{A}(u) < d \right\}$$

is arcwise connected. Similar conclusion if $u \geq 0$ is replaced by $u \leq 0$.

Proof: Since o is empty if $d \leq \lambda_1(m)$, we can assume from now on $d > \lambda_1(m)$. Using lemma (3.6), one constructs a weight $\hat{n} \in L^q(\partial\Omega)$ such that $\hat{n}^+ \neq 0, \lambda_1(\hat{n}) > d$ and $\hat{n} \leq m$. It suffices in this construction to take $\hat{n} = \varepsilon m^+ - m^-$ with $\varepsilon > 0$ to be sufficiently small. We then consider the manifold $M_{m,\hat{n}}$ and the sublevel set

$$\hat{O} := \left\{ u \in M_{m,n}; A(u) < d \right\}.$$

By Proposition (2.3), the restriction $\tilde{A}$ of A to $M_{m,\hat{n}}$ satisfies the Palais-Smale condition. Lemma (2.2) implies that any (nonempty) component of $\hat{O}$ contains a critical point of $\tilde{A}$ But the first two critical levels $\lambda_1(m), \lambda_1(\hat{n})$ of $\tilde{A}$ verify $\lambda_1(m) < d < \lambda_1(\hat{n})$ and consequently $\tilde{A}$ admits only one critical point in $\hat{O}$. We can conclude in this way that $\hat{O}$ is arcwise connected. Let now $u_1, u_2 \in O$. Since they are ≥ 0, they also belong to $\hat{O}$. Let γ be a patch in $\hat{O}$ from u_1, u_2 and consider the patch

$$\gamma_1(t) := \frac{|\gamma(t)|}{(\frac{1}{p}\int_{\partial\Omega} m|\gamma(t)|^p)^{\frac{1}{p}}}$$

By the choice of $\hat{n}$, we have

$$\int_{\partial\Omega} m|\gamma(t)|^p \, d\sigma \geq \int_{\partial\Omega} m(\gamma(t)^+ + \hat{n}(\gamma(t)^-)^p \, d\sigma = 1, \tag{3.12}$$

and consequently γ_1 is well defined patch in $M_{m,n}$, which clearly goes from u_1 to u_2 and is made of nonnegative functions. Moreover, by (3.12),

$$A(\gamma_1(t) = \frac{A(\gamma(t))}{\frac{1}{p}\int_{\partial\Omega} m|\gamma(t)|^p} \leq A(\gamma(t)) < d,$$

for all t, and we conclude that the patch γ_1 lies in O. This concludes the proof of Lemma (3.7) for O with $u \geq 0$. similar argument in the case $u \leq 0$

Lemma 3.8. There exist $u_1 \geq 0$ and $u_2 \leq 0$ in $M_{m,n}$ such that $\tilde{A}(u_1) < c(m,n)$ and $\tilde{A}(u_2) < c(m,n)$. Moreover, for any such choice of u_1, u_2, one has

$$c(m,n) = \inf_{\gamma \in \bar{\Gamma}} \max_{u \in \gamma([0,1])} \tilde{A}(u), \tag{3.13}$$

where

$$\bar{\Gamma} := \left\{\gamma \in C([0,1], M_{m,n}); \gamma(0) = u_2 \; and \; \gamma(1) = u_1\right\},$$

and c(m, n) is defined in (3.1).

Proof: Since $m^+ \neq 0$, one take $u_1 = \varphi_m$ and the inequality $\tilde{A}(u_1) < c(m,n)$ follows from Lemma (3.5). Similarly with $u_2 = -\varphi_n$. It remains to prove (3.13). Call $\bar{c}$ the right-hand side of (3.13). One clearly has $c(m,n) \leq \bar{c}$. To prove the converse inequality, let $\varepsilon > 0$ and take $\gamma_\varepsilon \in \Gamma$ such that

$$\max_{u \in \gamma_\in([0,1])} \tilde{A}(u) < c(m,n) + \varepsilon.$$

By Lemma (3.7), there exists a patch $\eta_1 \in M_{m,n}$ joining $\gamma_\varepsilon(1)$ and u_1, made of nonnegative functions and such that

$$\max_{u\in\eta_1([0,1])} \tilde{A}(u) < c(m,n)+\varepsilon.$$

Similarly there exists a patch $\eta_2 \in M_{m,n}$ joining $\gamma_\varepsilon(0)$ and u_2, made of nonpositive functions and such that

$$\max_{u\in\eta_2([0,1])} \tilde{A}(u) < c(m,n)+\varepsilon.$$

Gluing together $\eta_2, \gamma_\varepsilon$ and η_1, one gets a path in $M_{m,n}$ joining u_2 and u_1 and such that $\tilde{A}$ remains $< c(m,n)+\varepsilon$ along this path. This implies $\overline{c} \le c(m,n)+\varepsilon$. Since $\varepsilon > 0$ is arbitrary, the conclusion follows.

We are now in a position to give the proof of Theorem (3.1).

Proof of Theorem 3.1. By Lemma (3.5), one has $c(m,n) > \max\{\lambda_1(m), \lambda_1(n)\}$. To prove that $c(m,n)$ is an eigenvalue, we pick u_1, u_2 as in Lemma (3.8) and we will show that $\overline{c}$, the right-hand side of (3.13), is a critical values of $\tilde{A}$. By Proposition (3.2) $\tilde{A}$ satisfies the Palais-Smale condition. Thus the classical mountain pass theorem for a C^1 functional manifold (see Proposition (2.1) yields the conclusion.

It remains to show that there is no eigenvalue between $\max\{\lambda_1(m), \lambda_1(n)\}$ and $c(m,n)$. Assume by contradiction the existence of such an eigenvalue λ and let u be the corresponding eigenfucntion. We know that u change sign (since $\lambda > \max\{\lambda_1(m), \lambda_1(n)\}$); moreover

$$0 < \| u^+ \|_{1,p}^p = \lambda \int_{\partial\Omega} m(u^+)^p \, d\sigma, \quad 0 < \| u^- \|_{1,p}^p = \lambda \int_{\partial\Omega} n(u^-)^p \, d\sigma,$$

and we can normalize u so that $u \in M_{m,n}$. The function

$$u_1 := \frac{u^+}{\left(\frac{1}{p}\int_{\partial\Omega} m(u^+)^p \, d\sigma\right)^{\frac{1}{p}}}, \quad u_2 := \frac{-u^-}{\left(\frac{1}{p}\int_{\partial\Omega} n(u^-)^p \, d\sigma\right)^{\frac{1}{p}}},$$

belong to $M_{m,n}$, with $u_1 \ge 0, u_2 \le 0$. We will construct a path γ in $M_{m,n}$ joining u_1 and u_2 and such that $\tilde{A}(\gamma(t)) = \lambda$ for all $t \in [0,1]$. This will give a contradiction with $c(m,n) \le \max_{u\in\gamma([0,1])} \tilde{A}(u) = \lambda$. To construct γ we first go from u_1 to u by the path γ_1 such that $u \in \gamma([0,1])$

$$\gamma_1(t) := \frac{u^+ - tu^-}{(B_{m,n}(u^+ - tu^-))^{\frac{1}{p}}},$$

and then from u to u_2 by the path γ_2 such that

$$\gamma_2(t) := \frac{tu^+ - u^-}{(B_{m,n}(tu^+ - u^-))^{\frac{1}{p}}}.$$

It is easily to verified that the path constructed in this way is well defined and satisfies all the required conditions.

We are now in a position to give an application of Theorem (3.1) concerning the second positive eigenvalue $\lambda_2(m)$ of problem (2.2). In application, for $m = n$, we obtain another variational characterization for the $\lambda_2(m)$:

Corollary 3.9.

$$\lambda_2(m) = c(m,m) = \inf_{\gamma\in\Gamma_0} \max_{u\in\gamma([0,1])} \left(\frac{1}{p} \| u \|_{1,p}^p \right),$$

where

$$\Gamma_0 = \left\{ \gamma \in C([0,1], M_{m,m}) : \gamma(0) \leq 0 \ \ and \ \ \gamma(1) \geq 0 \right\},$$

$$M_{m,m} := \left\{ u \in W^{1,p}(\Omega); \frac{1}{p} \int_{\partial\Omega} m \,|\, u \,|^p \; d\sigma = 1 \right\},$$

(for example $\gamma(1) = \varphi_m$ and $\gamma(0) = -\varphi_n$).

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