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The Semi Orlicz Space of χ of Analytic

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Abstract

Let Γ denote the space of all entire sequences. Let Λ denote the space of all analytic sequences. This paper is to introduce a new class of sequence spaces namely the semi Orlicz space of χ of analytic. It is shown that the intersection of all semi Orlicz space of χ of analytic is semi Orlicz space of χ of analytic.

Keywords: χ-sequence, analytic sequence, Orlicz space, entire sequence, duals.

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1. Introduction

A complex sequence, whose k^{th} terms is x_k is denoted by $\{x_k\}$ or simply x Let w be the set of all sequences $x = (x_k)$ and ϕ be the set of all finite sequences. Let ℓ_∞, c, c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. In respect of ℓ_∞, c, c_0 we have $\| x \| = \sup_k | x_k |$, where $x = (x_k) \in c_0 \subset c \subset \ell_\infty$. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k | x_k |^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called entire sequence if $\lim_{k\to\infty} | x_k |^{1/k} = 0$. The vector space of all entire sequences will be denoted by $\Gamma.\chi$. was discussed in kamthan [19]. Matrix transformation involving χ were characterized by Sridhar [20] and Sirajiudeen [21]. Let χ be the set of all those sequences $x = (x_k)$ such that $(k! | x_k |)^{1/k} \to 0$ as $k \to \infty$. Then χ is a metric space with the metric

$$d(x,y)=\sup_k\left\{(k!\,|\,x_k-y_k\,|)^{1/k}:k=1,2,3,...\right\}$$

Orlicz [4] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [5] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p(1\le p<\infty)$. Subsequently different classes of sequence spaces defined by Parashar and Choudhary [6], Mursaleen et al.[7], Bektas and Altin [8], Tripathy et al. [9], Rao and Subramanian [10] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref [11].

Recall ([4], [11]) an Orlicz function is a function $M:[0,\infty)\to[0,\infty)$ which is continuous, non-decreasing and convex with $M(0)=0, M(x)>0$, for $x>0$ and $M(x)\to\infty$ and $x\to\infty$. If convexity of Orlicz funtion M is replaced by $M(x+y)\le M(x)+M(y)$ then this function is called modulus function, introduced by Nakano [18] and further discussed by Ruckle [12] and Maddox [13] and many others.

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K>0$, such that $M(2u)\le KM(u)(u\ge 0)$. The Δ_2-condition is equivalent to $M(\ell u)$, for all values of u and for $\ell>1$. Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M=\left\{x\in w:\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)<\infty, for some\, \rho>0\right\}. \tag{1}$$

The space ℓ_M with the norm

$$\|x\|=inf\left\{\rho>0:\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)\le 1\right\} \tag{2}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t)=t^p, 1\le p<\infty$, the space ℓ_M coincide with the classical sequence space ℓ_p. Given a sequence $x=\{x_k\}$ its n^{th} section is the sequence $x^{(n)}=\{x_1,x_2,...,x_n,0,0,...\}\delta^{(n)}=(0,0,...,1,0,0,...)$, 1 in the n^{th} place and zero's else where; and $s^{(k)}=(0,0,...,1,-1,0,...)$, 1 in the n^{th} place, -1 in the $(n+1)^{th}$ place and zero's else where. An FK-space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x)=x_k(k=1,2,3,...)$ are continuous. We recall the following definitions [see [15]].

An FK-space is a locally convex Frechet space which is made up of sequences and has the property that coordinate projections are continuous. An metric-space (X,d) is said to have AK (or sectional convergence) if and only if $d(x^{(n)},x)\to x$ as $n\to\infty$. [see [15]] The space is said to have AD (or) be an AD space if ϕ is dense in X. We note that AK implies AD by [14].

If X is a sequence space, we define

(i) $X^{'} =$ the continuous dual of X .

(ii) $X^{\alpha} = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty$, for each $x \in X\}$;

(iii) $X^{\beta} = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k$ is convergent, for each $x \in X\}$;

(iv) $X^{\gamma} = \left\{a = (a_k) : \sup_n | \sum_{k=1}^{n} a_k x_k | < \infty, \quad for\ each\ x \in X\right\}$;

(v) Let X be an FK-space $\supset \phi$. Then $X^{f} = \{ f(\delta^{(n)}) : f \in X^{'}\}$.

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called the $\alpha-$ (or Kothe-Toeplitz) dual of X, $\beta-$(or generalized Ko the-T oeplitz) dual of X, $\gamma-$dual of X . Note that $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$. If $X \subset Y$ then $Y^{\mu} \subset X^{\mu}$, for $\mu = \alpha, \beta$, or γ.

1.1 Lemma

(See (15, Theorem 7.27)). Let X be an FK-space $\supset \phi$. Then

(i) $X^{\gamma} \subset X^{f}$. (ii) If X has AK, $X^{\beta} = X^{f}$. (iii) If X has AD, $X^{\beta} = X^{\gamma}$.

2. Definitions and Prelimiaries

Let w denote the set of all complex double sequences $x = (x_k)_{k=1}^{\infty}$ and $M : [0, \infty) \to [0, \infty)$ be an Orlicz function, or a modulus function. Let

$$\chi_M = \left\{x \in w : lim_{k\to\infty} \left(M\left(\frac{(k!|x_k|)^{1/k}}{\rho}\right)\right) = 0\, for some\, \rho > 0\right),$$

$$\Gamma_M = \left\{x \in w : lim_{k\to\infty} \left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right) = 0\, for some\, \rho > 0\right) \text{ and}$$

$$\Lambda_M = \left\{x \in w : sup_k \left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right) < \infty\, for some\, \rho > 0\right)$$

The space χ_M is a metric space with the metric

$$d(x, y) = inf \left\{\rho > 0 : sup_k \left(M\left(\frac{(k!\,|x_k - y_k|)^{1/k}}{\rho}\right)\right) \leq 1\right\} \tag{3}$$

The space Γ_M and Λ_M is a metric space with the metric

$$d(x, y) = inf \left\{\rho > 0 : sup_k \left(M\left(\frac{|x_k - y_k|^{1/k}}{\rho}\right)\right) \leq 1\right\} \tag{4}$$

Because of the historical roots of summability in convergence, conservative space and matrices play a special role in its theory. However, the results seem mainly to depend on a weaker assumption, that the spaces be semi conservative. (See[15]). Snyder and Wilansky[1] introduced the concept of semi conservative spaces.

Snyder[3] studied the properties of semi conservative spaces. In the year 1996 the semi replete spaces were introduced by Chandrasekhara Rao and Srinivasalu [2]. K.Chandrasekhara Rao and N.Subramanian [17] and [22] introduced the concept of semi analytic spaces and the semi Orlicz space of analytic sequences. Recently N.Subramanian, B.C.Tripathy and C.Murugesan has [23] introduced the concept of the semi Orlicz space of $cs \cap d_1$.

In a similar way, in this paper we define semi Orlicz space of of analytic, and hence show that Γ_M is smallest semi Orlicz space of χ of analytic.

3. Main Results

3.1 Proposition

$\chi_M \subset \Gamma_M$

Proof. It is easy. Therefore omit the proof.

3.2 Proposition

χ_M has AK, where M is a modulus function.

Proof. Let $x = \{x_k\} \in \chi_M$, but then $\left\{ M\left(\frac{(k!|x_k|)^{1/k}}{p} \right) \right\} \in \chi$, and hence $\sup_{k \geq n+1} \left(M\left(\frac{(k!|x_k|)^{1/k}}{p} \right) \right) \to$ as $n \to \infty$. Therefore

$$d\left(x, x^{[n]}\right) = inf\left\{ \rho > 0 : sup_{k\geq n+1}\left(M\left(\frac{(k!\,|x_k|)^{1/k}}{\rho} \right) \right) \leq 1 \right\} \to 0 as n \to \infty \tag{5}$$

$\Rightarrow x^{[n]} \to x$ as $n \to \infty$, implying that χ_M has AK. This completes the proof.

3.3 Proposition

$(\chi_M)^\beta = \Lambda$.

Proof. Step 1: $\chi_M \subset \Gamma_M$ by Proposition 3.1.

$\Rightarrow (\Gamma_M)^\beta \subset (\chi^M)^\beta$. But $(\Gamma_M)^\beta = \Lambda$. [see [10]].

$$\Lambda \subset (\chi^M)^\beta \tag{6}$$

Step 2: Let $y \in (\chi^M)^\beta$. But $f(x) = \sum_{k=1}^{\infty} x_k y_k$, with $x \in \chi_M$ we recall that s^k has

$\left(\frac{1}{k!}\right)$ in the k^{th} place and zero's elsewhere, with $x = s^k, \left\{M\left(\frac{k!|x_k|)^{1/k}}{p}\right)\right\} = \left\{0,0,...,\left(M\left(\frac{(0)^{1/k}}{p}\right),0,...\right)\right\}$ which converges to zero. Hence $s^k \in \chi_M$. Hence $d(s^k,0)=1$. But $|y_k| \le \|f\| d(s^k,0) < \infty$ for all k. Thus (y_k) is a bounded sequence and hence an analytic sequence. In other words $y \in \Lambda$.

$$(\chi_M)^\beta \subset \Lambda \tag{7}$$

From (6) and (7) we obtain $(\chi_M)^\beta = \Lambda$. This completes the proof.

3.4 Lemma

[15, Theorem 8.6.1] $Y \supset X \Leftrightarrow Y^f \subset X^f$ where X is an AD-space and Y an FK-space.

3.5 Proposition

Let Y be any FK-space $\supset \phi$. Then $Y \supset \chi_M$ if and only if the sequence $s^{(k)}$ is weakly analytic

Proof: The following implications establish the result.

$Y \supset \chi_M \Leftrightarrow Y^f \subset (\chi_M)^f$, since χ_M has AD by Lemma 3.4.

$\Leftrightarrow Y^f \subset \Lambda$, since $(\chi_M)^f = \Lambda$.

$\Leftrightarrow$ for each $f \in Y'$, the topological dual of Y. $f(s^{(k)}) \in \Lambda$.

$\Leftrightarrow f(s^{(k)})$ is analytic.

$\Leftrightarrow s^{(k)}$ is weakly analytic. This completes the proof.

4. Properties of Semi Orlicz Space of χ of Analytic

4.1 Definition

An FK-space X is called "semi Orlicz space of χ of analytic" if its dual $(X)^f \subset \Lambda$. In other words X is semi Orlicz space of χ of analytic if $f(s^{(k)}) \in \Lambda$ forall $f \in (X)'$ for each fixed k.

4.2 Example

χ_M is semi Orlicz space of χ of analytic. Indeed, if χ_M is the space of all Orlicz sequence of χ, then by Lemma 4.3 $(\chi_M)^f = \Lambda$.

4.3 Lemma
$(\chi_M)^f = \Lambda$.

Proof: $(\chi_M)^\beta = \Lambda$ by Theorem 3.3. But (χ_M) has AK by Proposition 3.2. Hence $(\chi_M)^\beta = (\chi_M)^f$. Therefore $(\chi_M)^f = \Lambda$ This completes the proof. We recall

4.4 Lemma
(See 15, Theorem 4.3.7) Let z be a sequence. Then (z^β, P) is an AK space with $P = (P_k : k = 0,1,2,...)$, where $P_0(x) = {}^{\sup}_{m} | \sum_{k=1}^{m} z_k x_k |, P_n(x) = | x_n |$. For any k such that $z_k \neq 0, P_k$ may be omitted. If $z \in \phi, P_0$ may be omitted.

4.5 Proposition
Let z be a sequence z^β is semi Orlicz space of χ of analytic if and only if z is Λ.

Proof: Step 1: Suppose that z^β is semi Orlicz space of χ of analytic. z^β has AK by Lemma 4.4. Therefore $Z^{\beta\beta} = (z^\beta)^f$ by Theorem 7.2.7 of Wilansky [15]. So Z^β is semi Orlicz space of χ of analytic if and only if $z^{\beta\beta} \subset \Lambda$. But then $zk \in z^{\beta\beta} \subset \Lambda$. Hence z is Λ.

Step 2: Conversely, suppose that z is Λ. Then $z^\beta \supset \{\Lambda\}^\beta$ and $z^{\beta\beta} \subset \{\Lambda\}^{\beta\beta} = \Gamma^\beta = \Lambda$, because $\Lambda^\beta = \Gamma$. But $(z^\beta)^f = z^{\beta\beta}$. Hence $(z^\beta)^f \subset \Lambda$. Therefore z^β is semi Orlicz space of χ of analytic. This completes the proof.

4.6 Proposition
Every semi Orlicz space of χ of analytic contained Γ_M.

Proof: Let X be any semi Orlicz space of χ of analytic. Hence $(X)^f \subset \Lambda$. Therefore $f(s^{(k)}) \in \Lambda$ forall $f \in (X)'$. So, $\left\{s^{(k)}\right\}$ is weakly analytic with respect to X. Hence $X \supset \chi_M$ by Proposition 3.5. But $\chi_M \subset \Gamma_M$. Hence $X \subset \Gamma_M$ This completes the proof.

4.7 Proposition
The intersection of all semi Orlicz space of χ of analytic $\left\{X_n : n = 1,2,...\right\}$ is semi Orlicz space of χ of analytic.

Proof: Let $X = \bigcap_{n=1}^{\infty} X_n$. Then X is an FK-space which contains ϕ. Also every

$f \in (X)'$ can be written as $f = g_1 + g_2 + ... + g_m$, where $g_k \in (X_n)'$ for some n and for $1 \le k \le m$. But then $f(s^k) = g_1(x^k) + g_2(s^k) + ... + g_m(s^k)$. Since $X_n (n = 1, 2, ...)$ are semi Orlicz space of χ of analytic, it follows that $g_i(s^k) \in \Lambda$ for all $i = 1, 2, ...m$. Therefore $f(s^k) \in \Lambda$ forall k and for all f. Hence X is semi Orlicz space of χ of analytic. This completes the proof.

4.8 Proposition

The intersection of all semi Orlicz space χ of analytic is Γ_M.

Proof: Let I be the intersection of all semi Orlicz space of χ of analytic. By Proposition 4.5 we see that the intersection

$$I \subset \bigcap \left\{ z^\beta : z \in \Lambda \right\} = \{\Lambda\}^\beta = \Gamma = \Gamma_M. \quad (8)$$

By Proposition 4.7 it follows that I is semi Orlicz space of χ of analytic. Consequently

$$\Gamma_M \subset I \text{ (by Proposition 4.6)} \quad (9)$$

From (8) and (9) we get $I = \Gamma_M$. This completes the proof.

4.9 Corollary

The smallest semi Orlicz space of χ of analytic is Γ_M.

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