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# Generalized Past Entropy in Survival Analysis

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**Abstract**

In many realistic situations uncertainty is not necessarily related to the future but can also refer to the past. In the context of information theory, measure of uncertainty in past life time distributions has been proposed by Crescemzo and Longobardi $[5]$. In this paper, we introduce Varma's information measure for some past life time distributions. Based on this measure, some characterization results for some lifetime distributions are addressed and new classes of life time distributions are defined. Various properties of these classes are also given.

**Key Words:** Shannon's entropy, generalized entropy, past residual entropy, lifetime distributions.
**AMS Subject classification:** 60E15, 62N05, 90B25, 94A17, 94A24.

## 1. Introduction

Let $X$ be a continuous random variable with probability density function $f(x)$, the basic measure of uncertainty is given by Shannon $[18]$ and is defined as

$$H(X)=-\int_0^\infty f(x)\log f(x)dx \tag{1}$$

Varma generalize (1) and defined the entropy of order $\alpha$ and type $\beta$ as

$$H_v(\alpha,\beta,X)=\frac{1}{\beta-\alpha}\log\int_0^\infty f^{\alpha+\beta-1}(x)dx \quad \text{for } \beta-1<\alpha<\beta,\ \ \beta\geq 1 \tag{2}$$

For various properties and applications of Varma's entropy, one should refer to $[21]$. It must be noted that

$\lim_{\beta=1,\alpha\to1} H_v(\alpha,\beta,X)=-\int_0^\infty f(x)\log f(x)\,dx$ which is Shannon,s entropy given in (1).

Ebrahimi $[7]$ defined the uncertainty of residual life time distribution $H(X;t)$ by truncating the distributions below some point $t$, of a component as

$$H(X;t)=-\int_t^\infty \frac{f(x)}{R(t)}\log\frac{f(x)}{R(t)}dx \tag{3}$$

$$=1-\frac{1}{R(t)}\int_t^\infty \log r_F(x)\,dx$$

where $R(t)$ is the survival function and $r_F(t)=\frac{f(t)}{R(t)}$ is the hazard rate function.

It is reasonable to presume that in many realistic situations uncertainty is not necessarily related to future but can also refer to the past . Based on this idea, Crescemzo and Longobardi $[5]$ have introduced the concept of past entropy over $(0,t)$. If $X$ denotes the life time of a component or of living organism , then the past entropy of $X$ is defined as

$$H^0(X;t)=-\int_0^t \frac{f(x)}{F(t)}\log\frac{f(x)}{F(t)}dx$$

$$=1-\frac{1}{F(t)}\int_0^t f(x)\log\tau(x)\,dx \tag{4}$$

where $F(t)$ is the cumulative distribution function and $\tau(t)=\frac{f(t)}{F(t)}$ is the reversed hazard function or reversed failure rate of $X$ .

In view of the growing importance of the concept of the reversed hazard function,

Chandra and Roy $[4]$examined some result on implicative relationship in the context of the monotonic behavior of reversed hazard function.

In the present paper, we study the characterization results of the past lifetime distributions by using the following generalized information measure

$$H_v^0(X;t)=\frac{1}{\beta-\alpha}\log\frac{\int_0^t f^{\alpha+\beta-1}(x)\,dx}{F^{\alpha+\beta-1}(t)}\ ,\ \beta-1<\alpha<\beta\ ,\ \beta\geq1 \tag{5}$$

Note that, as $\beta=1,\alpha\to1$, equation (5) reduces to (4). Again as $\beta=1,\alpha\to1$ and $t\to\infty$, (5) reduces to (1).

## 2. Some Characterization Results

**A. Continuous Distribution:** Let $X$ be a continuous non-negative random variable having distribution function $F(t)=P(T\leq t)$. A natural question in this context is whether $H_v^0(X;t)$ uniquely determines the distribution. In the following theorem, we prove that $H_v^0(X;t)$ determines $F(t)$ uniquely.

**Theorem 1:** Let $X$ be a continuous non-negative random variable with distribution function $F(t)$ and an increasing generalized past entropy, $H_v^0(X;t)$. Then $H_v^0(X;t)$ uniquely determines $F(t)$.

**Proof:** From (5), we have

$$\frac{\int_0^t f^{\alpha+\beta-1}(x)dx}{F^{\alpha+\beta-1}(t)}=\exp\left\{(\beta-\alpha)H_v^0(X;t)\right\}$$

or $$\int_0^t f^{\alpha+\beta-1}(x)dx=F^{\alpha+\beta-1}(t)\exp\left\{(\beta-\alpha)H_v^0(X;t)\right\} \tag{6}$$

Differentiating both sides with respect $t$, we have

$$\tau^{\alpha+\beta-1}(t)=(\alpha+\beta-1)\tau(t)\exp\left\{(\beta-\alpha)H_v^0(X;t)\right\}$$
$$+(\beta-\alpha)H'^0_v(X;t)\exp\left\{(\beta-\alpha)H_v^0(X;t)\right\}$$

where $\tau(t)=\dfrac{f(t)}{F(t)}$ is the reversed hazard function.

Hence for fixed $t>0$, $\tau(t)$ is a solution of

$$\begin{aligned}\eta(x)&=x^{\alpha+\beta-1}-(\alpha+\beta-1)x\exp\left\{(\beta-\alpha)H_v^0(X;t)\right\}\\&-(\beta-\alpha)H'^0_v(X;t)\exp\left\{(\beta-\alpha)H_v^0(X;t)\right\}=0\end{aligned} \tag{7}$$

Differentiating both sides with respect to $x$, we have

$$\eta'(x)=(\alpha+\beta-1)x^{\alpha+\beta-2}-(\alpha+\beta-1)\exp\left\{(\beta-\alpha)H_v^0(X;t)\right\}$$

Note that $\eta'(x)=0$, gives

$$x=\exp\left\{\left(\frac{\beta-\alpha}{\alpha+\beta-2}\right)H_v^0(X;t)\right\}=x_t$$

Also, $\eta''(x)=(\alpha+\beta-1)(\alpha+\beta-2)x^{\alpha+\beta-3}$

**Two cases arises**

**Case I:** Let $\alpha+\beta>2$, then $\eta''(x_t)>0$, thus $\eta(x)$ attains the minimum at $x_t$.

Also $\eta(0)<0$ and $\eta(\infty)=\infty$. Further $\eta(x)$ decreases for $0<x<x_t$ and hence increases for $x>x_t$. So, $x=\tau(t)$ is the unique solution to $\eta(x)=0$.

**Case II:** Let $\alpha+\beta<2$, then $\eta''(x)<0$, thus $\eta(x)$ attains the maximum at $x_t$.

Also $\eta(0)<0$ and $\eta(\infty)=-\infty$. Further $\eta(x)$ increases for $0<x<x_t$ and hence decreases for $x>x_t$. So, $x=\tau(t)$ is the unique solution to $\eta(x)=0$.

Combining both the cases, we conclude that $H_v^0(X;t)$ uniquely determines $\tau(t)$ and hence $F(t)$.

**Theorem 2:** The uniform distribution over $(a,b),\ a<b$ is characterized by increasing past entropy $H_v^0(X;t)=\dfrac{\alpha+\beta-2}{\alpha-\beta}\log(t-a),\ t>a$ with $\alpha+\beta<2$

**Proof:** In case of uniform distribution over $(a,b),\ a<b$, we have

$H_v^0(X;t)=\dfrac{\alpha+\beta-2}{\alpha-\beta}\log(t-a)$, which is increasing in $t>a$.

Differentiating with respect to $t$, we have

$$H'^0_v(X;t)=\frac{\alpha+\beta-2}{(\alpha-\beta)(t-a)}$$

Also, $x_t=\exp\left\{\left(\dfrac{\beta-\alpha}{\alpha+\beta-2}\right)H_v^0(X;t)\right\}=(t-a)^{-1}$

Substituting the value of $H'^0_v(X;t)$ and $x_t$ in (7), we get

$\eta(x_t)=0$.

**B. Discrete Distribution:** Let $X$ is a discrete random variable taking the values $x_0,x_1,...,x_n$ with the probabilities $p_0,p_1,...,p_n$. The past uncertainty of discrete random lifetime distribution is defined as

$$H^0(P;j)=-\sum_{k=0}^{j}\frac{p_k}{P(j)}\log\left(\frac{p_k}{P(j)}\right) \tag{8}$$

where $P(j)=\sum_{k=0}^{j}p_k$ is the discrete distribution function of $X$.

The generalized past residual entropy for the discrete case is defined as

$$H_v^0(P;j)=\frac{1}{\beta-\alpha}\log\left\{\sum_{k=0}^{j}\left(\frac{p_k^{\alpha+\beta-1}}{P^{\alpha+\beta-1}(j)}\right)\right\},\ \beta-1<\alpha<\beta,\ \beta\geq 1 \tag{9}$$

For $\beta=1,\alpha\to 1$, equation (9) reduces to equation (8).

**Theorem 3:** If $X$ has a discrete distribution function $F(t)$ with support $(0,1,2,...,n)$ and an increasing generalized past residual entropy $H_v^0(P;j)$. Then $H_v^0(P;j)$ uniquely determine $F(j)$.

**Proof:** We have

$$H_v^0(P;j)=\frac{1}{\beta-\alpha}\log\left\{\sum_{k=0}^{j}\left(\frac{p_k^{\alpha+\beta-1}}{P^{\alpha+\beta-1}(j)}\right)\right\},\ \beta-1<\alpha<\beta,\ \beta\geq 1$$

or $\sum_{k=0}^{j} p_k^{\alpha+\beta-1} = p^{\alpha+\beta-1}(j)\exp\{(\beta-\alpha)H_v^0(P;j)\}$ (10)

For $j+1$, we have

$$\sum_{k=0}^{j+1} p_k^{\alpha+\beta-1} = p^{\alpha+\beta-1}(j+1)\exp\{(\beta-\alpha)H_v^0(P;j+1)\} \quad (11)$$

Subtracting (11) from (10), writing $P_{j+1} = P(j+1) - P(j)$, we have

$$\exp\{(\beta-\alpha)H_v^0(P;j+1)\} = (1-\lambda_j)^{\alpha+\beta-1} + \lambda_j^{\alpha+\beta-1}\exp\{(\beta-\alpha)H_v^0(P;j)\}$$

where $\lambda_j = \frac{P(j)}{P(j+1)} \in (0,1)$. It can be noted that for a fixed $x>0$, $x=\lambda_j$ is a solution to

$$\xi(x) = (1-x)^{\alpha+\beta-1} + x^{\alpha+\beta-1}\exp\{(\beta-\alpha)H_v^0(P;j)\} - \exp\{(\beta-\alpha)H_v^0(P;j+1)\} = 0 \quad (12)$$

Differentiating both sides with respect to $x$, we get

$$\xi'(x) = -(\alpha+\beta-1)(1-x)^{\alpha+\beta-2} + (\alpha+\beta-1)x^{\alpha+\beta-2}\exp\{(\beta-\alpha)H_v^0(P;j)\}$$

Note that $\xi'(x)=0$, gives

$$x = \left[1+\exp\left\{\left(\frac{\beta-\alpha}{\alpha+\beta-2}\right)H_v^0(P;j)\right\}\right]^{-1} = x_j$$

Also,

$$\xi''(x) = (\alpha+\beta-1)(\alpha+\beta-2)(1-x)^{\alpha+\beta-3} + (\alpha+\beta-1)(\alpha+\beta-2)x^{\alpha+\beta-3}\exp\{(\beta-\alpha)H_v^0(P;j)\}$$

Further, from (12) we have

$\xi(0) = 1-\exp\{(\beta-\alpha)H_v^0(P;j+1)\}$, $\xi(1) = \exp\{(\beta-\alpha)H_v^0(P;j)\} - \exp\{(\beta-\alpha)H_v^0(P;j+1)\}$

Two cases rises:

**Case I:** Let $\alpha+\beta>2$, then $\xi''(x)\geq 0$. Again

$\xi'(x)>0$ *if* $x>x_j$

$\xi'(x)=0$ *if* $x=x_j$

$\xi'(x)<0$ *if* $x<x_j$

so that $\xi(x)=0$ has a unique solution if $\xi(x_j)=0$.

**Case II:** Let $\alpha+\beta<2$, then $\xi''(x)\leq 0$. Again

$\xi'(x)>0$ *if* $x<x_j$

$\xi'(x)=0$ *if* $x=x_j$

$\xi^{/}(x)<0 \;\; if \;\; x>x_j$

so that $\xi(x)=0$ has a unique solution if $\xi(x_j)=0$.

Combining both the cases, we conclude that $H_v^0(P;j)$ uniquely determines $\lambda_j$, which in turns determines $F(j)$ uniquely.

**Theorem 4:** The uniform distribution with support $\{0,1,2,...,n\}$ is characterized by increasing discrete past entropy $H_v^0(P;j)=\dfrac{\alpha+\beta-2}{\alpha-\beta}\log(j+1),\;\; j=0,1,2,...,n.$ with $\alpha+\beta<2$

**Proof:** In Case of uniform distribution with support $\{0,1,2,...,n\}$, we have

$H_v^0(P;j)=\dfrac{\alpha+\beta-2}{\alpha-\beta}\log(j+1),\;\; j=0,1,2,...,n$ which is increasing in $j$

Also,

$$x_j=\left[1+\exp\left\{\left(\frac{\beta-\alpha}{\alpha+\beta-2}\right)H_v^0(P;j)\right\}\right]^{-1}$$

or $x_j=\left(\dfrac{j+1}{j+2}\right)$.

Therefore,

$\xi(x_j)=0$.

Thus $\xi(x)=0$ has unique solution given by $x=x_j$. But $\lambda_j$ is the solution of (12).

Hence $\lambda_j$ is the unique solution to $\xi(x)=0$. This shows that the distribution is discrete uniform and the theorem is proved.

## 3. A New Class of Life Time Distribution

In this section, we propose a new type of class of life time distribution based on the generalized past entropy.

**Definition :** A non negative random variable $X$ is said to have increasing uncertainty of life (IUL) if $H^0(X;t)$ is increasing in $t\geq 0$.

**Definition :** A non negative random variable $X$ is said to have increasing uncertainty of life of order $\alpha$ and type $\beta$ (IUL($\alpha,\beta$)) if $H_v^0(X;t)$ is increasing in $t\geq 0$.

**Lemma 1 :** For a non-negative continuous random variable $X$, define $Z=aX+b$, where $a>0, b\geq 0$ are constant. Then, for $t>b$

$$H_v^0(Z;t)=\frac{2-\alpha-\beta}{\beta-\alpha}\log(a)+H_v^0\left(X;\frac{t-b}{a}\right).$$

**Proof :** We have,

$$H_v^0(X\,;\,t)=\frac{1}{\beta-\alpha}\log\frac{\int_0^t f^{\alpha+\beta-1}(x)dx}{F^{\alpha+\beta-1}(t)}\ ,\ \beta-1<\alpha<\beta\ ,\ \beta\geq 1.$$

Also, $Z=aX+b$

There fore,

$$H_v^0(Z\,;t)=\frac{2-\alpha-\beta}{\beta-\alpha}\log(a)+H_v^0\left(X\,;\frac{t-b}{a}\right)$$ , which proves the lemma.

**Theorem 5:** If $X$ is (IUL($\alpha,\beta$)), then

(I) $\tau(t)\geq(\alpha+\beta-1)^{\frac{1}{\alpha+\beta-2}}\exp\left\{\left(\frac{\beta-\alpha}{\alpha+\beta-2}\right)H_v^0(X\,;t)\right\}\quad if\quad \alpha+\beta>2$

(II) $\tau(t)\leq(\alpha+\beta-1)^{\frac{1}{\alpha+\beta-2}}\exp\left\{\left(\frac{\beta-\alpha}{\alpha+\beta-2}\right)H_v^0(X\,;t)\right\}\quad if\quad \alpha+\beta<2$

**Proof:** Since $X$ is (IUL($\alpha,\beta$)), therefore

$H'^0_v(X\,;t)\geq 0$, which gives

$\tau(t)\geq(\alpha+\beta-1)^{\frac{1}{\alpha+\beta-2}}\exp\left\{\left(\frac{\beta-\alpha}{\alpha+\beta-2}\right)H_v^0(X\,;t)\right\}\quad if\quad \alpha+\beta>2$ and

$\tau(t)\leq(\alpha+\beta-1)^{\frac{1}{\alpha+\beta-2}}\exp\left\{\left(\frac{\beta-\alpha}{\alpha+\beta-2}\right)H_v^0(X\,;t)\right\}\quad if\quad \alpha+\beta<2$. Hence the theorem is proved.

**Corollary:** Let $F(t)$ be a (IUL($\alpha,\beta$)), then

(I) $F(t)\leq\exp\left\{-\int_t^\infty(\alpha+\beta-1)^{\frac{1}{\alpha+\beta-2}}\exp\left(\left(\frac{\beta-\alpha}{\alpha+\beta-2}\right)H_v^0(X\,;u)\right)du\right\}\ if\ \alpha+\beta>2.$

(II) $F(t)\geq\exp\left\{-\int_t^\infty(\alpha+\beta-1)^{\frac{1}{\alpha+\beta-2}}\exp\left(\left(\frac{\beta-\alpha}{\alpha+\beta-2}\right)H_v^0(X\,;u)\right)du\right\}\ if\ \alpha+\beta<2$

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