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The Orbit of Probability Density Functions by the Deformed Tent Map

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Abstract

Let φ be deformed tent map, A_φ be Perron-Frobenius operator, f be an arbitrary probability density function on closed interval [0, 1]. We show the orbit of ${A_\varphi}^m f$ and obtain the main theroem. We draw the graphs of the orbit of ${A_\varphi}^m f$.

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Introduction

In [1], Prof. Kawamura studied a linear operator A in a Banach lattice β which was similar to the Perron-Frobenius operator A_φ in $L^1(X)$ associated with a $MWnL\varphi$. Kawamura showed the behavior of the orbit of a positive unit vector with respect to the iteration of A. The main result was to give two spaces M and N, which are subspaces of β and satisfy the following convergence property:

$$\lim_{m\to\infty} ||{A_\varphi}^m x - e||_1 = 0$$

In paticular, let φ be Tent map τ, for every probablity density function f on [0, 1], the following holds

$$\lim_{m\to\infty} ||{A_\varphi}^m f - \chi_{[0,1]}||_1 = 0.$$

In the present paper, let φ be deformed tent map defined by

$$\varphi(x) = \begin{cases} x & \text{if } x \in \left[0,\ \frac{1}{4}\right] \\ 3x - \frac{1}{2} & \text{if } x \in \left[\frac{1}{4},\ \frac{1}{2}\right] \\ -2x + 2 & \text{if } x \in \left[\frac{1}{2},\ 1\right] \end{cases} \tag{0.1}$$

we study the orbits of ${A_\varphi}^m f$, the probablity density functions by deformed tent map. The background information and theorems are introduced in Section 1, the main result is shown in Section 2, and the graphs of the orbits of ${A_\varphi}^m f$ are drawn in Section 3.

1. The Background

Definition 1.1. Let X be a compact metric space and f be a R-valued Lebesgue integrable function on X. Let μ be Lebesgue measure. If the following (1) and (2) are held at the same time, f is called a probability density function.

(1) $\forall x \in X,\ f(x) = 0,$ (2) $\int_X f(x) d\mu(x) = 1$

Definition 1.2. Let X be a compact metric space'Cμ be the Lebesgue measure'DWe denote by $L^1(X,\ \mu)$ the set of all integrable functions f on X. Namely

$$L^1(X,\ \mu) = \left\{ f \;\middle|\; \int_X |f|\, d\mu < +\infty \right\} \tag{1.2}$$

Furthermore, we denote by PDF(X) the set of probability density functions on X

$$\mathrm{PDF}(X) = \left\{ f \;\middle|\; f \geq 0,\ \int_X f\, d\mu = 1 \right\} \tag{1.3}$$

Obviously we know

$$\mathrm{PDF}(X) \subset L^1(X,\ \mu).$$

Thus, for $f \in \mathrm{PDF}(X)$, by the one-time synthesis of chaos dynamical system map φ, we see the change of f. In fact, by

$$\int_X \eta(\varphi(x)) f(x)\, d\mu(x) \quad (\eta \in L^\infty(X,\ \mu)) \tag{1.4}$$

the observation is possible. Here, we introduce the Perron–Frobenius operator A_φ : $f \mapsto A_\varphi f$. A_φ can meet

$$\int_X \eta(\varphi(x)) f(x)\, d\mu(x) = \int_X \eta(x)(A_\varphi f)(x)\, d\mu(x)$$

A_φ has the following qualities.

Lemma 1.3 *Let X be compact metric space and φ be a continuous map, $\varphi : X \to X$. Then A_φ is operator of* PDF(X) *into* PDF(X). *Moreover for $\alpha,\ \beta \in \mathrm{R}$, $f_1,\ f_2 \in L^1(X, \mu)$, we have*

$$A_\varphi(\alpha f_1 + \beta f_2) = \alpha A_\varphi f_1 + \beta A_\varphi f_2.$$

By this, the Perron–Frobenius operator is bounded linear operator of $L^1(X,\ \mu)$ into $L^1(X,\ \mu)$. (See [5])

Theorem 1.4. *Let continuous map $\varphi : [0,\ 1] \to [0,\ 1]$. Furthermore, $\exists c \in (0,\ 1)$, if there exists the following map*'G(See [5])

$$\begin{aligned} &(1)\ \ \varphi_1 : [0,\ c] \to [0,\ 1], \ \ \textit{strictly monotone increasing'Cbijective map} \\ &\qquad \varphi_1,\ \varphi_1^{-1} \textit{are absolutely continuous} \\ &(2)\ \ \varphi_2 : [c,\ 1] \to [0,\ 1], \ \ \textit{strictly monotone decreasing'Cbijective map} \\ &\qquad \varphi_2,\ \varphi_2^{-1} \textit{are absolutely continuous} \end{aligned} \tag{1.5}$$

For $\forall f \in$ PDF$([0,\ 1])$ *we have*

$$(A_\varphi f)(x) = \left|\frac{d{\varphi_1}^{-1}}{dx}(x)\right| f({\varphi_1}^{-1}(x)) + \left|\frac{d{\varphi_2}^{-1}}{dx}(x)\right| f({\varphi_2}^{-1}(x)) \tag{1.6}$$

Definition 1.5. Let $\{f_n\}_{n=1}^{\infty} \subset L^1(X,\ \mu)$, $f \in L^1(X,\ \mu)$.

(1) f_n converges pointwisely to f,

$$\textit{for}\ \ \forall x \in X,\ \ \lim_{n\to\infty} f_n(x) = f(x)$$

Namely, we have

$$\forall x \in X,\ \forall \varepsilon > 0,\ \exists N = N(x,\ \varepsilon) \in \mathrm{N}\ \text{ s.t. }\ \forall n \geq N \ \ \Longrightarrow\ \ |f_n(x) - f(x)| < \varepsilon.$$

(2) f_n converges almost everywhere to f, there exist zero set N in X'Cf_n converges pointwisely to f in $X \setminus N$. Namely, we have

$$\exists N \subset X,\ \mu(N) = 0\ \text{ s.t. }\ \lim_{n\to\infty} f_n(x) = f(x)\ \ (x \notin N).$$

(3) f_n converges to f in L^1, we have

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0.$$

Definition 1.6. An arbitrary f in $L^1(X,\ \mu)$ is corresponding to real number $\|f\|$, when the following (1)'‘(4) meets'C$\|\cdot\|$ is called Norm. Let $f,\ g \in L^1(X,\ \mu),\ \alpha \in$ R'DThen we have

(1) $\|f\| \geq 0$,

(2) $\|f\| = 0 \iff f = 0\ \ (a.e.)$,

(3) $\|\alpha f\| = |\alpha|\|f\|$,

(4) $\|f + g\| \leq \|f\| + \|g\|$

In $L^1(X,\ \mu)$, we define

$$\|f\|_1 = \int_X |f|\, d\mu$$

Then $\|\cdot\|_1$ is Norm on $L^1(X,\ \mu)$.

2. The Orbit of Probability Density Functions by the Deformed Tent Map

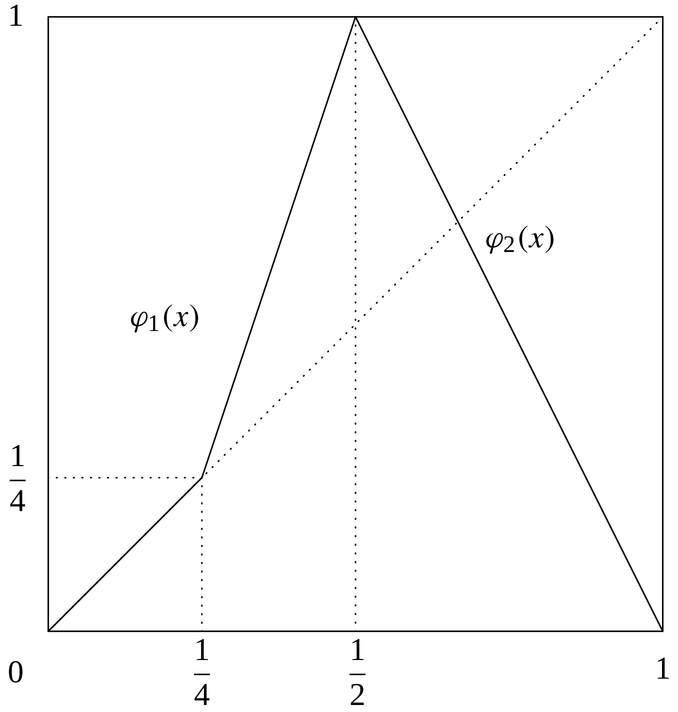
In this section φ is the deformed tent map on closed interval $[0,\ 1]$ defined by

$$\varphi(x) = \begin{cases} x & \text{if } x \in \left[0,\ \frac{1}{4}\right] \\ 3x - \frac{1}{2} & \text{if } x \in \left[\frac{1}{4},\ \frac{1}{2}\right] \\ -2x + 2 & \text{if } x \in \left[\frac{1}{2},\ 1\right] \end{cases} \tag{2.7}$$

According to this definition, if

$$\varphi_1(x) = \begin{cases} x & \text{if } x \in \left[0,\ \frac{1}{4}\right] \\ 3x - \frac{1}{2} & \text{if } x \in \left[\frac{1}{4},\ \frac{1}{2}\right] \end{cases} \quad \varphi_2(x) = -2x + 2 \ \text{ on } \ x \in \left[\frac{1}{2},\ 1\right]$$

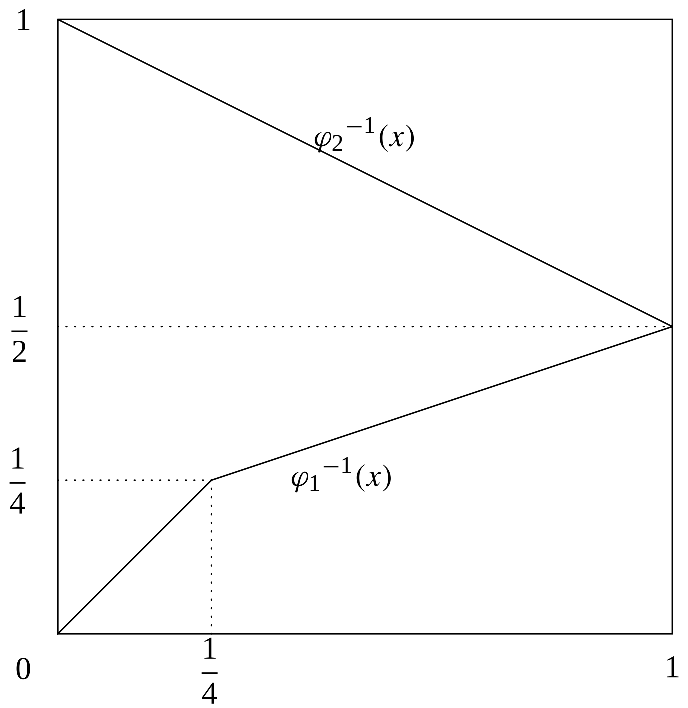
The graph is in following



φ_1 is a monotone increasing bijective mapping'Cand φ_2 is a monotone decreasing bijective mapping'C Then we can consider inverse mapping ${\varphi_1}^{-1}$, ${\varphi_2}^{-1}$

$$ {\varphi_1}^{-1}(x) = \begin{cases} x & \text{if } x \in \left[0,\ \frac{1}{4}\right] \\ \frac{1}{3}x + \frac{1}{6} & \text{if } x \in \left[\frac{1}{4},\ 1\right] \end{cases} \quad {\varphi_2}^{-1}(x) = -\frac{1}{2}x + 1 \ \text{ on } \ x \in [0,\ 1] $$

The graph is in the following



We can see that ${\varphi_1}^{-1}$ is a monotone increasing bijective mapping'C${\varphi_2}^{-1}$ is a monotone decreasing bijective mapping'D

Corollary 2.1. *In Theorem 1.4, let φ be the deformed tent map, then*

$$(A_\varphi f)(x) = \begin{cases} f(x) + \frac{1}{2} f(1 - \frac{1}{2}x) & x \in [0, \frac{1}{4}] \\ \frac{1}{3} f(\frac{1}{3}x + \frac{1}{6}) + \frac{1}{2} f(1 - \frac{1}{2}x) & x \in [\frac{1}{4}, 1] \end{cases}$$

Next, the set of inverse mapping $[0, \frac{1}{4}]$ is defined as the following.

$$I_0 = I_{0(1)} = [a_{0(1)},\ b_{0(1)}] = \left[0,\ \frac{1}{4}\right] = {\varphi_1}^{-1}(I_0)$$

$$I_1 = I_{1(2)} = [a_{1(2)},\ b_{1(2)}] = \left[\frac{7}{8},\ 1\right] = {\varphi_2}^{-1}(I_0)$$

$$I_{2(2,1)} = [a_{2(2,1)},\ b_{2(2,1)}] = \left[\frac{11}{24},\ \frac{1}{2}\right] = {\varphi_1}^{-1}(I_1)$$

$$I_{2(2,2)} = [a_{2(2,2)},\ b_{2(2,2)}] = \left[\frac{1}{2},\ \frac{9}{16}\right] = {\varphi_2}^{-1}(I_1)$$

$$I_{3(2,1,1)} = [a_{3(2,1,1)},\ b_{3(2,1,1)}] = \left[\frac{23}{72},\ \frac{24}{72}\right] = {\varphi_1}^{-1}(I_{2(2,1)})$$

$$I_{3(2,1,2)} = [a_{3(2,1,2)},\ b_{3(2,1,2)}] = \left[\frac{36}{48},\ \frac{37}{48}\right] = {\varphi_2}^{-1}(I_{2(2,1)})$$

$$I_{3(2,2,1)} = [a_{3(2,2,1)},\ b_{3(2,2,1)}] = \left[\frac{16}{48},\ \frac{17}{48}\right] = {\varphi_1}^{-1}(I_{2(2,2)})$$

$$I_{3(2,2,2)} = [a_{3(2,2,2)},\ b_{3(2,2,2)}] = \left[\frac{23}{32},\ \frac{24}{32}\right] = {\varphi_1}^{-2}(I_{2(2,2)})$$

$$\vdots$$

And, for $k = 2,\ 3,\ \cdots\cdots$ when $\{I_{k(2,p_2,p_3,\cdots,p_k)}\}_{(p_2,p_3,\cdots,p_k)\in\{1,2\}^{k-1}}$ is decided recursively, $\{I_{k+1,(2,p_2,p_3,\cdots,p_k,p_{k+1})}\}_{(p_2,p_3,\cdots,p_k,p_{k+1})\in\{1,2\}^k}$ is defined as the following'D

$$\begin{cases} I_{k+1,(2,p_2,p_3,\cdots,p_k,1)} = {\varphi_1}^{-1}(I_{k,(2,p_2,p_3,\cdots,p_k)}) \\ I_{k+1,(2,p_2,p_3,\cdots,p_k,2)} = {\varphi_2}^{-1}(I_{k,(2,p_2,p_3,\cdots,p_k)}) \end{cases} \tag{2.8}$$

From this on, show as the following.

$$P = (p_1, p_2, p_3, \cdots, p_k)$$
$$\pi_0 = \{1\}, \quad \pi_1 = \{2\}, \quad \pi_k = \{2\} \times \{1, 2\}^{k-1}$$
$$I_{k,P} = [a_{k,P}, b_{k,P}]$$

Lemma 2.2. *For*

$$\bigcup_{k=0}^{\infty} \bigcup_{P \in \pi_k} \{I_{k,P}\} \tag{2.9}$$

the following can be held.

(1) *(9) is mutal prime in the sense of Lebesgue measure.*

(2)

$$\mu\left(\bigcup_{k=0}^{\infty} \bigcup_{P \in \pi_k} \{I_{k,P}\}\right) = \mu([0, 1]) = 1 \tag{2.10}$$

(3) *If* $f \in \mathrm{PDF}([0,\ 1])$ *let*

$$f_l = \sum_{k=0}^{l} \sum_{P \in \pi_k} f \cdot \chi_{I_{k,\ P}} \tag{2.11}$$

Then f_l converges to f almost everywhere and converges to f in L^1.

Proof. (1) We show it by deviding into two steps'D

Step 1. To $k \geq 2$'C$\mu(I_{k,\ P} \cap I_{k,\ P'}) = 0$ ($P \neq P'$)'D
We show it by mathematical induction. When $k = 2$'C

$$\begin{aligned}\mu(I_{2,\ P} \cap I_{2,\ P'}) &= \mu(I_{2(2,1)} \cap I_{2(2,2)}) \\ &= \mu\left(\left[\frac{11}{24},\ \frac{1}{2}\right] \cap \left[\frac{1}{2},\ \frac{9}{16}\right]\right) = \mu\left(\left\{\frac{1}{2}\right\}\right) \\ &= 0.\end{aligned}$$

For a certain k, suppose $I_{k,\ P} \cap I_{k,\ P'}$ ($P \neq P'$) has at most one common point.
When $k+1$'C$I_{k+1,P} = I_{k+1,(2,p_2,p_3,\cdots,p_k,p_{k+1})}$ and $I_{k+1,P'} = I_{k+1,(2,p_2{}',p_3{}',\cdots,p_k{}',p_{k+1}{}')}$ there are three kinds of probability'DIn the following let $P \neq P'$'D

Case 1. When $p_{k+1} = p_{k+1}{}' = 1$, ${\varphi_1}^{-1}$ is monotone increasing, if we note it bijective mapping.

$$\begin{aligned}\mu(I_{k+1,\ P} \cap I_{k+1,\ P'}) &= \mu({\varphi_1}^{-1}(I_{k,\ P}) \cap {\varphi_1}^{-1}(I_{k,\ P'})) \\ &= \mu({\varphi_1}^{-1}(I_{k,\ P} \cap I_{k,\ P'})) \\ &\leq \mu({\varphi_1}^{-1}(\{a_l\})) = 0.\end{aligned} \tag{2.12}$$

Case 2. When $p_{k+1} = {p_{k+1}}' = 2$, ${\varphi_2}^{-1}$ is monotone decreasing, if we note it bijective mapping. **Case 2.** and **Case 1.** are the same'D

Case 3. When $p_{k+1} \neq {p_{k+1}}'$, We may suppose without loss of generality that $p_{k+1} = 1$, ${p_{k+1}}' = 2$.

$$I_{k+1,\ P} \cap I_{k+1,\ P'} = [a_{k+1,P},\ b_{k+1,P}] \cap [a_{k+1,P'},\ b_{k+1,P'}]$$

By the assumption

$$a_{k+1,P} < b_{k+1,P} \leq \frac{1}{2} \leq a_{k+1,P'} < b_{k+1,P'}.$$

And then'C$\mu(I_{k+1,\ P} \cap I_{k+1,\ P'}) \leq \mu(\{\frac{1}{2}\}) = 0$.

By the above'Cto $k \geq 2$'C$\mu(I_{k,\ P} \cap I_{k,\ P'}) = 0$'D

Step 2. For $\forall s,\ t \geq 0,\ s \neq t$'C$\mu(I_{s,\ P} \cap I_{t,\ P'}) = 0$'D
We show $I_{s,\ P} \cap I_{t,\ P'} = \emptyset$ by reduction to absurdity'D

Suppose $I_{s,\ P} \cap I_{t,\ P'} \neq \emptyset$
We may suppose without loss of generality that $s < t$'Cfor $l \geq 1$ let $t = s + l$, and $x \in I_{s,\ P} \cap I_{t,\ P'}$

$$\varphi^s(x) \in [a_0,\ b_0] = I_0 \tag{2.13}$$

On the one hand'C$\exists P'' = (p_1,\ p_2,\ p_3, \cdots,\ p_{t-s})$,Å

$$\varphi^s(x) \in [a_{l,P''},\ b_{l,P''}] = I_{l,\ P''} \tag{2.14}$$

By $l \geq 1$, then $I_{l,\ P''} \neq I_0$'DHence,it is contradiction'D
Therefor $I_{s,\ P} \cap I_{t,\ P'} = \emptyset$

By **Step 1** and **Step 2** (1) is shown.

(2) We devide into two steps'DFirst'Cwe note $\forall k \in \mathtt{N}$, $I_{k,\ P} \subset [0,\ 1]$'Cand

$$\mu(I_0) = \frac{1}{4},\ \ \mu(I_1) = \frac{1}{8}$$

Step 1. When $k \geq 2$

$$\mu\left(\bigcup_{P \in \pi_k} I_{k,\ P}\right) = \frac{1}{8} \cdot \left(\frac{5}{6}\right)^{k-1} \tag{2.15}$$

can be shown by mathematical induction'DWhen $k = 2$

$$\mu\left(\bigcup_{P\in\pi_k} I_{2,\ P}\right) = \mu(I_{2(2,1)} \cup I_{2(2,2)}) = \mu(I_{2(2,1)}) + \mu(I_{2(2,1)})$$
$$= \mu\left(\left[\frac{11}{24},\ \frac{1}{2}\right]\right) + \mu\left(\left[\frac{1}{2},\ \frac{9}{16}\right]\right)$$
$$= \frac{1}{24} + \frac{1}{16} = \frac{1}{8}\left(\frac{1}{3} + \frac{1}{2}\right)$$
$$= \frac{1}{8} \cdot \frac{5}{6}.$$

for a certain k'Csuppose (15) is held.
When $k + 1$, we use the result of (1)

$$\mu\left(\bigcup_{P\in\pi_{k+1}} I_{k+1,\ P}\right) = \sum_{P\in\pi_{k+1}} \mu(I_{k+1,\ P})$$
$$= \sum_{P\in\pi_k, p_{k+1}=1} \mu(I_{k+1,\ P}) + \sum_{P\in\pi_k, p_{k+1}=2} \mu(I_{k+1,\ P})$$
$$= \sum_{P\in\pi_k} \mu({\varphi_1}^{-1}(I_{k,\ P})) + \sum_{P\in\pi_k} \mu({\varphi_2}^{-1}(I_{k,\ P}))$$
$$= \sum_{P\in\pi_k} \mu\left(\left[\frac{1}{3}a_{k,P} + \frac{1}{6},\ \frac{1}{3}b_{k,P} + \frac{1}{6}\right]\right)$$
$$+ \sum_{P\in\pi_k} \mu\left(\left[-\frac{1}{2}b_{k,P} + 1,\ -\frac{1}{2}a_{k,P} + 1\right]\right)$$
$$= \frac{1}{3}\sum_{P\in\pi_k} \mu(I_{k,\ P}) + \frac{1}{2}\sum_{P\in\pi_k} \mu(I_{k,\ P})$$
$$= \frac{5}{6}\sum_{P\in\pi_k} \mu(I_{k,\ P})$$
$$= \frac{5}{6}\mu\left(\bigcup_{P\in\pi_k} I_{k,\ P}\right)$$
$$= \frac{5}{6} \cdot \frac{1}{8} \cdot \left(\frac{5}{6}\right)^{k-1}$$
$$= \frac{1}{8} \cdot \left(\frac{5}{6}\right)^{k}.$$

When $k + 1$ it is held'D

By the above, for all natural number k, (15)is held'D

Step 2. We prove

$$\mu\left(\bigcup_{k=1}^{\infty}\bigcup_{P\in\pi_k} I_{k,\ P}\right) = \mu(I_0)+\mu(I_1)+\sum_{k=2}^{\infty}\mu\left(\bigcup_{P\in\pi_k} I_{k,\ P}\right)$$
$$= \frac{1}{4}+\frac{1}{8}+\sum_{k=2}^{\infty}\frac{1}{8}\left(\frac{5}{6}\right)^{k-1}$$
$$= \frac{1}{4}+\frac{1}{8}\cdot\frac{1}{1-5/6} = \frac{1}{4}+\frac{3}{4} = 1$$

On the one hand, since $\mu([0,\ 1]) = 1$'C

$$\mu\left(\bigcup_{k=0}^{\infty}\bigcup_{P\in\pi_k} I_{k,\ P}\right) = \mu([0,\ 1]) = 1$$

is held.

(3) First, prove $f_l \longrightarrow f \quad a.e. \quad .$

Let $N = [0,\ 1] - \bigcup_{k=0}^{\infty}\bigcup_{P\in\pi_k} I_{k,\ P}$.

Then $\mu(N) = \mu\left([0,\ 1] - \bigcup_{k=0}^{\infty}\bigcup_{P\in\pi_k} I_{k,\ P}\right) = 0$

$\forall x \in [0,\ 1] \quad$ if $x \notin N, \quad \exists l \quad x \in \bigcup_{k=0}^{l}\bigcup_{P\in\pi_k} I_{k,\ P}$ furthermore $\exists j,\ P' \quad x \in I_{j,P'}$ then

$$f_l(x) = \sum_{k=0}^{l}\sum_{P\in\pi_k} f\cdot\chi_{I_{k,\ P}}(x) = f\cdot\chi_{I_{j,\ P'}}(x) = f(x).$$

Next, prove $f_l \longrightarrow f \quad$ in L^1

$$f_l = \sum_{k=0}^{l}\sum_{P\in\pi_k} f\cdot\chi_{I_{k,\ P}} \tag{2.16}$$

for each $l = 0, 1,\ 2,\ \cdots$ then $f_l \le f$, and when $l \to \infty$'C$f_l \uparrow f$, so by the convergence theorem of Lebesgue

$$\lim_{l\to\infty}\int_{[0,\ 1]} f_l\,d\mu = \int_{[0,\ 1]}\lim_{l\to\infty} f_l\,d\mu = \int_{[0,\ 1]} f\,d\mu = 1 \tag{2.17}$$

is held. By this we know

$$\lim_{l\to\infty} ||f - f_l||_1 = 0.$$

Next, we observe the orbit $\{A_\varphi{}^m f\}_{m=0}^{\infty}$ for $\forall\ f \in \mathrm{PDF}([0,\ 1])$, A_φ is a Perron–Frobenius operator.

Theorem 2.3. *If φ is the deformed tent map, A_φ is a Perron–Frobenius operator, $f \in \mathrm{PDF}([0,\ 1])$'Clet*

$$f_\infty(x) = \sum_{k=0}^{\infty} \sum_{P\in\pi_k} \left|((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1})'(x)\right| f((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1}(x))\chi_{I_0}(x) \tag{2.18}$$

At this time, we have the following:

(1) *(18) is convergence of almost everywhere and convergence in L^1.*

(2) $A_\varphi{}^m f$ *converge to* f_∞ *in* L^1.

In order to prove the theorem, we prove several lemmas at first.

Lemma 2.4. *For φ is the deformed tent map, for $m \in \mathrm{N}$. Then*

$$\begin{aligned} A_\varphi{}^m(f \cdot \chi_{I_{k,\ P}})(x) = &\left|((\varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1})'(x)\right| \\ &f((\varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1}(x))\chi_{\varphi^m(I_{k,\ P})}(x) \end{aligned} \tag{2.19}$$

Where $P \in \pi_k$

Proof. We show it by mathematical induction. When $m = 1$, by Theorem 1.4

$$\begin{aligned} A_\varphi(f \cdot \chi_{I_{k,\ P}})(x) = &|(\varphi_1{}^{-1})'(x)| f(\varphi_1{}^{-1}(x))\chi_{I_{k,\ P}}(\varphi_1{}^{-1}(x)) \\ &+ |(\varphi_2{}^{-1})'(x)| f(\varphi_2{}^{-1}(x))\chi_{I_{k,\ P}}(\varphi_2{}^{-1}(x)) \end{aligned} \tag{2.20}$$

Case 1. When $\varphi_1{}^{-1}(x) \in I_{k,\ P}$ and $\varphi_2{}^{-1}(x) \notin I_{k,\ P}$, then
$\chi_{I_{k,\ P}}(\varphi_2{}^{-1}(x)) = 0$, and

$$\varphi_1{}^{-1}(x) \in I_{k,\ P} \iff x \in \varphi_1(I_{k,\ P}) = I_{k-1,\ P}\ \ (p_k = 1) \iff I_{k,\ P} \subset \left[0,\ \frac{1}{2}\right]$$

so'C

$$(20) = |(\varphi_1{}^{-1})'(x)| f(\varphi_1{}^{-1}(x))\chi_{\varphi_1(I_{k,\ P})}(x) = |(\varphi_1{}^{-1})'(x)| f(\varphi_1{}^{-1}(x))\chi_{\varphi(I_{k,\ P})}(x).$$

Case 2. When $\varphi_2{}^{-1}(x) \in I_{k,\ P}$ and $\varphi_1{}^{-1}(x) \notin I_{k,\ P}$, then
$\chi_{I_{k,\ P}}(\varphi_1{}^{-1}(x)) = 0$, and

$$\varphi_2{}^{-1}(x) \in I_{k,\ P} \iff x \in \varphi_2(I_{k,\ P}) = I_{k-1,\ P}\ \ (p_k = 2) \iff I_{k,\ P} \subset \left[\frac{1}{2},\ 1\right]$$

so'C

$$(20) = |({\varphi_2}^{-1})'(x)|f({\varphi_2}^{-1}(x))\chi_{\varphi_2(I_{k,\ P})}(x) = |({\varphi_2}^{-1})'(x)|f({\varphi_2}^{-1}(x))\chi_{\varphi(I_{k,\ P})}(x).$$

Case 3. When ${\varphi_1}^{-1}(x) \notin I_{k,\ P}$ and ${\varphi_2}^{-1}(x) \notin I_{k,\ P}$, then $\chi_{I_{k,\ P}}({\varphi_1}^{-1}(x)) = \chi_{I_{k,\ P}}({\varphi_2}^{-1}(x)) = 0$ and

$$\text{Right of} \quad (20) = 0.$$

On the one hand $x \notin \varphi_1(I_{k,\ P})$ and $x \notin \varphi_2(I_{k,\ P}) = x \notin \varphi(I_{k,\ P})$. Hence

$$\text{Left of} \quad (20) = 0.$$

By the above, when $m = 1$, we can show in the form of (20)'D

For a certain m, we suppose (20) is held'Cwhen $m + 1$

$$\begin{aligned}
(A_{\varphi}{}^{m+1}(f \cdot \chi_{I_{k,\ P}}))(x) &= (A_{\varphi}(A_{\varphi}{}^{m}(f \cdot \chi_{I_{k,\ P}})))(x) \\
&= |({\varphi_1}^{-1})'(x)|(A_{\varphi}{}^{m}(f \cdot \chi_{I_{k,\ P}}))({\varphi_1}^{-1}(x)) \\
&\quad + |({\varphi_2}^{-1})'(x)|(A_{\varphi}{}^{m}(f \cdot \chi_{I_{k,\ P}}))({\varphi_2}^{-1}(x)) \\
&= |({\varphi_1}^{-1})'(x)|((\varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1})'({\varphi_1}^{-1}(x)) \qquad (2.21)\\
&\quad f((\varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1}({\varphi_1}^{-1}(x))\chi_{\varphi^m(I_{k,\ P})}({\varphi_1}^{-1}(x)) \\
&\quad + |({\varphi_2}^{-1})'(x)|((\varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1})'({\varphi_2}^{-1}(x)) \\
&\quad f((\varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1}({\varphi_2}^{-1}(x))\chi_{\varphi^m(I_{k,\ P})}({\varphi_2}^{-1}(x))
\end{aligned}$$

Case 3-1. When ${\varphi_1}^{-1}(x) \in \varphi^m(I_{k,\ P})$ and ${\varphi_2}^{-1}(x) \notin \varphi^m(I_{k,\ P})$, then $\chi_{\varphi^m(I_{k,\ P})}({\varphi_2}^{-1}(x)) = 0$, and

$$ {\varphi_1}^{-1}(x) \in \varphi^m(I_{k,\ P}) \iff x \in \varphi_1(\varphi^m(I_{k,\ P})) \iff \varphi^m(I_{k,\ P}) \subset \left[0,\ \frac{1}{2}\right]$$

At this time,

$$(\varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1}({\varphi_1}^{-1}(x)) = (\varphi_1 \circ \varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1}(x)$$

We derive each side by x

$$({\varphi_1}^{-1})'(x)((\varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1})'({\varphi_1}^{-1}(x)) = ((\varphi_1 \circ \varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1})'(x) \qquad (2.22)$$

So, we have

$$\begin{aligned}
(21) = |((\varphi_1 \circ \varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1})'(x)| \\
f((\varphi_1 \circ \varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1}(x))\chi_{\varphi^{m+1}(I_{k,\ P})}(x) \qquad (2.23)
\end{aligned}$$

Case 3-2. When ${\varphi_2}^{-1}(x) \in \varphi^m(I_{k,\ P})$ and ${\varphi_1}^{-1}(x) \notin \varphi^m(I_{k,\ P})$, then $\chi_{\varphi^m(I_{k,\ P})}({\varphi_1}^{-1}(x)) = 0$, and

$$ {\varphi_2}^{-1}(x) \in \varphi^m(I_{k,\ P}) \iff x \in \varphi_2(\varphi^m(I_{k,\ P})) \iff \varphi^m(I_{k,\ P}) \subset \left[\frac{1}{2},\ 1\right]$$

At this time,

$$(\varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1}(\varphi_2{}^{-1}(x)) = (\varphi_2 \circ \varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1}(x)$$

We derive each side by x

$$(\varphi_2{}^{-1})'(x)((\varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1})'(\varphi_2{}^{-1}(x)) = ((\varphi_2 \circ \varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1})'(x) \tag{2.24}$$

So, we have

$$\begin{aligned}(21) = |((\varphi_2 \circ \varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1})'(x)| \\ f((\varphi_2 \circ \varphi_{p_{k-m+1}} \circ \cdots \circ \varphi_{p_{k-1}} \circ \varphi_{p_k})^{-1}(x))\chi_{\varphi^{m+1}(I_{k,\ P})}(x)\end{aligned} \tag{2.25}$$

Case 3-3. When $\varphi_1{}^{-1}(x) \notin \varphi^m(I_{k,\ P})$ and $\varphi_2{}^{-1}(x) \notin \varphi^m(I_{k,\ P})$,
then $\chi_{\varphi^m(I_{k,\ P})}(\varphi_1{}^{-1}(x)) = \chi_{\varphi^m(I_{k,\ P})}(\varphi_2{}^{-1}(x)) = 0$ and

$$\text{Right of} \quad (21) = 0$$

On the one hand $x \notin \varphi_1(\varphi^m(I_{k,\ P}))$ and $x \notin \varphi_2(\varphi^m(I_{k,\ P})) \Longrightarrow x \notin \varphi(\varphi^m(I_{k,\ P}))$. Hence

$$\text{Left of} \quad (21) = 0$$

by (23)'C(25) we have

$$\begin{aligned}(A_\varphi{}^{m+1}(f \cdot \chi_{I_{k,\ i}}))(x) = ((\varphi_{p_{m+1}} \circ \varphi_{p_m} \circ \cdots \circ \varphi_{p_1})^{-1})'(x) \\ f((\varphi_{p_{m+1}} \circ \varphi_{p_m} \circ \cdots \circ \varphi_{p_1})^{-1}(x))\chi_{\varphi^{m+1}(I_{k,\ P})}(x).\end{aligned}$$

When $m + 1$, (19) is held.

By the above, for all natural number m (19) is held.

Corollary 2.5. *In Lemma 2.4, especially when* $m = 1$

$$A_\varphi(f \cdot \chi_{I_0})(x) = (f \cdot \chi_{I_0})(x) \tag{2.26}$$

$$A_\varphi(f \cdot \chi_{I_{1(2)}})(x) = |(\varphi_2^{-1})'(x)| f(\varphi_2^{-1}(x))\chi_{I_0}(x) \tag{2.27}$$

Proof. First, we show (26). For $I_0 \subset [0, \frac{1}{2}]$, $\varphi(I_0) = I_0$, and $x \in [0, \frac{1}{4}]$ $\varphi_1(x) = x$, so $\varphi_1{}^{-1}(x) = x$, and $(\varphi_1{}^{-1})'(x) = x$, Then

$$A_\varphi(f \cdot \chi_{I_0})(x) = |(\varphi_1{}^{-1})'(x)| f(\varphi_1{}^{-1}(x))\chi_{I_0}(x) = (f \cdot \chi_{I_0})(x).$$

Next, we show (27). $I_{1(2)}0 \subset [\frac{1}{2}, 1]$, $\varphi(I_{1(2)}0) = I_0$, so

$$A_\varphi(f \cdot \chi_{I_{1(2)}})(x) = |(\varphi_2{}^{-1})'(x)| f(\varphi_2{}^{-1}(x))\chi_{I_0}(x).$$

Corollary 2.6. *In Lemma 2.4, especially when $m = k$*

$$A_{\varphi}{}^{k}(f \cdot \chi_{I_{k,\ P}})(x) = \left|((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1})'(x)\right| f((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1}(x))\chi_{I_0}(x) \quad (2.28)$$

By this, for $\forall m \geq k$

$$A_{\varphi}{}^{m}(f \cdot \chi_{I_{k,\ P}})(x) = A_{\varphi}{}^{k}(f \cdot \chi_{I_{k,\ P}})(x) \quad (2.29)$$

Proof. First, we show (28).

$$\varphi^{k}(I_{k,P}) = \varphi^{k}(\varphi^{-1}(I_{k-1,P})) = \varphi^{k-1}(I_{k-1,P}) = \cdots = \varphi(I_1) = \varphi(\varphi^{-1}(I_0)) = I_0$$

So, by Lemma 2.4.

$$\begin{aligned} A_{\varphi}^{k}(f \cdot \chi_{I_{k,P}})(x) = & \left|((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1})'(x)\right| \\ & f((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1}(x))\chi_{\varphi^{k}(I_{k,P})}(x) \\ & = \left|((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1})'(x)\right| \\ & f((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1}(x))\chi_{I_0}(x) \end{aligned}$$

Next, we show (29). Let

$$(g \cdot \chi_{I_0})(x) = \left|((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1})'(x)\right| f((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1}(x))\chi_{I_0}(x)$$

From (28)

$$\begin{aligned} A_{\varphi}{}^{m}(f \cdot \chi_{I_{k,\ P}})(x) &= A_{\varphi}{}^{m-k}(A_{\varphi}{}^{k}(f \cdot \chi_{I_{k,\ P}}))(x) \\ &= A_{\varphi}{}^{m-k}(g \cdot \chi_{I_0})(x) \\ &= (g \cdot \chi_{I_0})(x) \\ &= A_{\varphi}{}^{k}(f \cdot \chi_{I_{k,\ P}})(x). \qquad q.e.d \end{aligned}$$

Corollary 2.7. *If φ is the deformed tent map, A_{φ} is Perron–Frobenius operator, $f \in$ PDF$([0,\ 1])$'Cwhen $\forall m \geq l$, we have*

$$A_{\varphi}{}^{m}(\sum_{k=0}^{l} \sum_{P \in \pi_k} f \cdot \chi_{I_{k,\ P}})(x) = \sum_{k=0}^{l} \sum_{P \in \pi_k} \left|((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1})'(x)\right| f((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1}(x))\chi_{I_0}(x) \quad (2.30)$$

Proof. By Corollary 2.5 and Corollary 2.6, it is clear.

Proof of Theorem 2.3. (1)

By (3) of Lemma 2.2 and Lebesegue convergence theorem

$$\begin{aligned}1 = ||f||_1 &= \int_{[0,1]} \sum_{k=0}^{\infty} \sum_{P \in \pi_k} f \cdot \chi_{I_{k,P}} d\mu \\ &= \sum_{k=0}^{\infty} \sum_{P \in \pi_k} \int_{[0,1]} f \cdot \chi_{I_{k,P}} d\mu \\ &= \lim_{l \to \infty} \sum_{k=0}^{l} \sum_{P \in \pi_k} \int_{[0,1]} f \cdot \chi_{I_{k,P}} d\mu \\ &= \lim_{l \to \infty} \int_{[0,1]} \sum_{k=0}^{l} \sum_{P \in \pi_k} f \cdot \chi_{I_{k,P}} d\mu.\end{aligned}$$

So if let

$$g_l(x) = \sum_{k=0}^{l} \sum_{P \in \pi_k} \left| ((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1})'(x) \right| \\ f((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1}(x)) \chi_{I_0}(x)$$

When $m \geq l$,

$$\begin{aligned}\int_{[0,1]} g_l d\mu &= \int_{[0,1]} A_{\varphi}{}^m \left(\sum_{k=0}^{l} \sum_{P \in \pi_k} f \cdot \chi_{I_{k,\ P}} \right) d\mu \\ &= \int_{[0,1]} \sum_{k=0}^{l} \sum_{P \in \pi_k} f \cdot \chi_{I_{k,\ P}} d\mu \\ &\longrightarrow \int_{[0,1]} f \cdot \chi_I d\mu \quad (l \to \infty).\end{aligned}$$

By this, we know that

$$\lim_{l,l' \to \infty} ||g_l - g_{l'}||_1 = 0.$$

So $\{g_l\}_{l=1}^{\infty}$ is sequence of Cauchy. Hence

$$\exists f_\infty = \lim_{l \to \infty} g_l \quad s.t. \quad ||f_\infty - g_l|| \longrightarrow 0 \tag{2.31}$$

and we choose partial sequence $\{g_{l_p}\}_{p=1}^{\infty}$ of $\{g_l\}$,

$$\{g_{l_p}\}_{p=1}^{\infty} \longrightarrow f_\infty \quad (a.e.)$$

and $\{g_l\}$ is monotone increasing sequence, so

$$g_l \longrightarrow f_\infty \quad (a.e.)$$

That is

$$f_\infty(x) = \lim_{l\to\infty} \sum_{k=0}^{l} \sum_{P\in\pi_k} \left|((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1})'(x)\right| f((\varphi_{p_1} \circ \varphi_{p_2} \circ \cdots \circ \varphi_{p_k})^{-1}(x))\chi_{I_0}(x) \quad (2.32)$$

(2). Given $\varepsilon > 0$. By Lemma 2.2. (3),

$$\exists l \in \mathrm{N} \quad ||f - f_l|| < \frac{\varepsilon}{2}$$

and for this l, by (3.1)

$$||f_\infty - g_l|| < \frac{\varepsilon}{2}$$

And when $m \geq l$,

$$\begin{aligned} ||f_\infty - A_\varphi{}^m f||_1 &\leq ||f_\infty - s_l||_1 + ||s_l - A_\varphi{}^m f||_1 \\ &< \frac{\varepsilon}{2} + ||A_\varphi{}^m f_l - A_\varphi{}^m f||_1 \\ &= \frac{\varepsilon}{2} + ||A_\varphi{}^m (f_l - f)||_1 \\ &= \frac{\varepsilon}{2} + ||f_l - f||_1 \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned} \quad (2.33)$$

3. The graph of orbit of probability density functions by the deformed tent map 1

In this section, we will use Mathematica 5.2 to draw the orbit graphs of $A_\varphi{}^n f \longrightarrow f_\infty$ in L^1 for $\forall\ f \in PDF$ where φ is the deformed tent map 1.

Let $f(x) = 4x^3$ again, since $\int_{[0,\ 1]} 4x^3\, d\mu = 1$, then $f \in PDF$. By Corollary 2.1

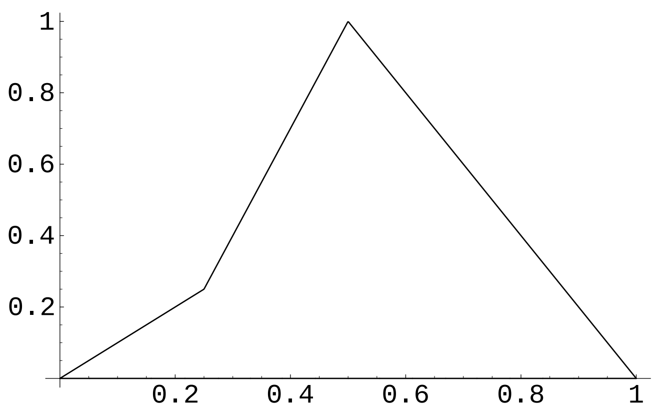
$$f_1(x) = (A_\varphi f)(x) = \begin{cases} f(x) + \frac{1}{2} f(1 - \frac{1}{2}x) & x \in [0, \frac{1}{4}] \\ \frac{1}{3} f(\frac{1}{3}x + \frac{1}{6}) + \frac{1}{2} f(1 - \frac{1}{2}x) & x \in [\frac{1}{4}, 1] \end{cases}$$

$$f_2(x) = (A_\varphi{}^2 f)(x) = (A_\varphi f_1)(x) = \begin{cases} f_1(x) + \frac{1}{2} f_1(1 - \frac{1}{2}x) & x \in [0, \frac{1}{4}] \\ \frac{1}{3} f_1(\frac{1}{3}x + \frac{1}{6}) + \frac{1}{2} f_1(1 - \frac{1}{2}x) & x \in [\frac{1}{4}, 1] \end{cases}$$

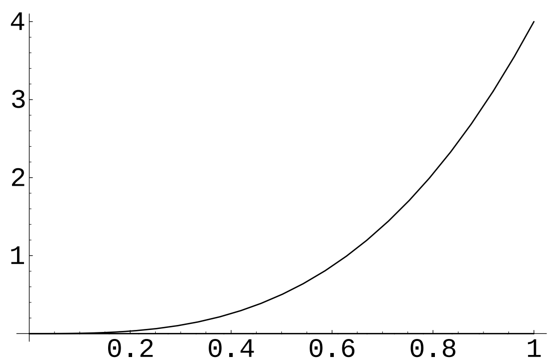
$$\vdots$$

$$f_n(x) = (A_\varphi{}^n f)(x) = (A_\varphi f_{n-1})(x) = \begin{cases} f_{n-1}(x) + \frac{1}{2} f_{n-1}(1 - \frac{1}{2}x) & x \in [0, \frac{1}{4}] \\ \frac{1}{3} f_{n-1}(\frac{1}{3}x + \frac{1}{6}) + \frac{1}{2} f_{n-1}(1 - \frac{1}{2}x) & x \in [\frac{1}{4}, 1] \end{cases}$$

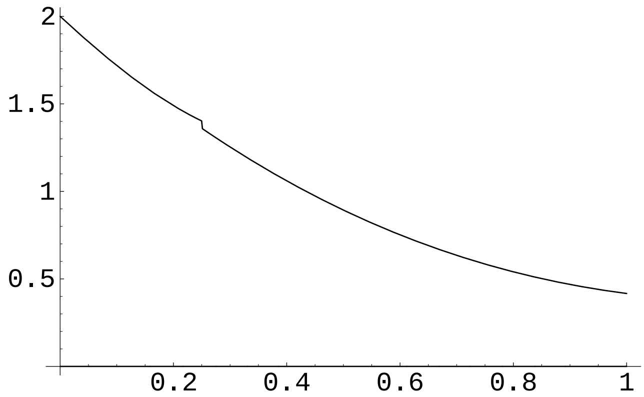
Then graphs are in the following.



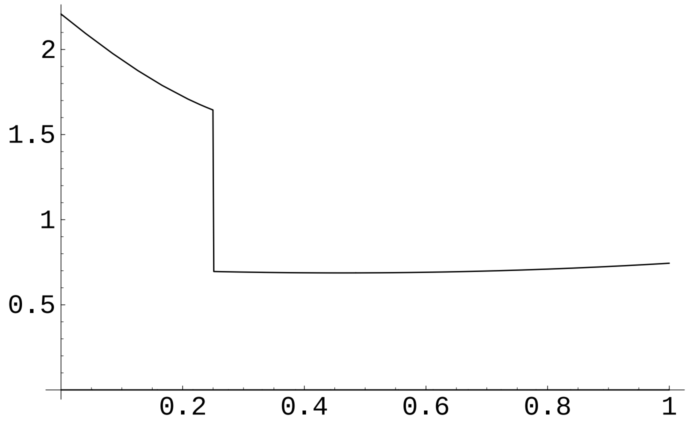
– Graph of $\varphi(x)$ –



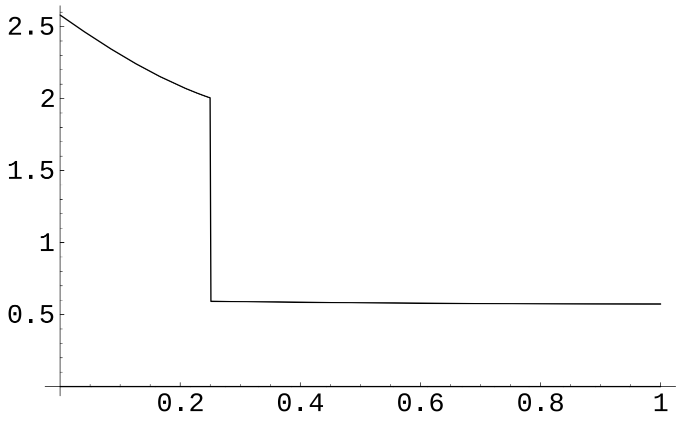
– Graph of $f(x)$ –



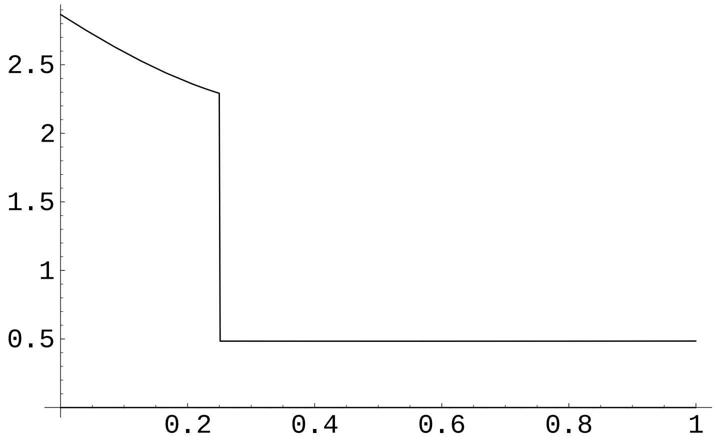
– Graph of $(A_\varphi f)(x)$ –



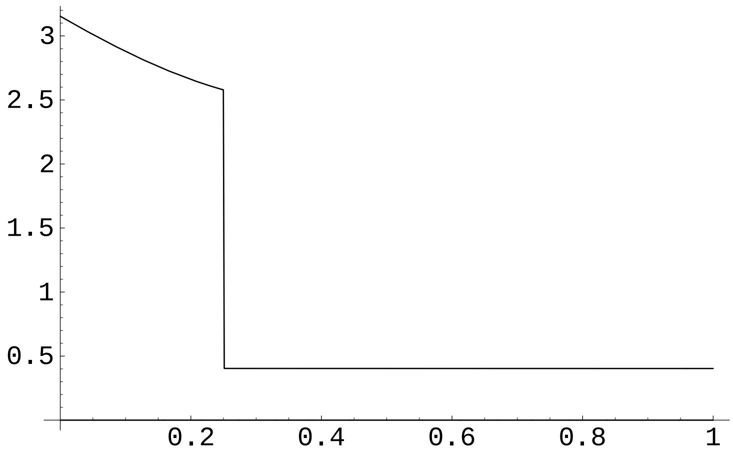
– Graph of $(A_\varphi{}^2 f)(x)$ –



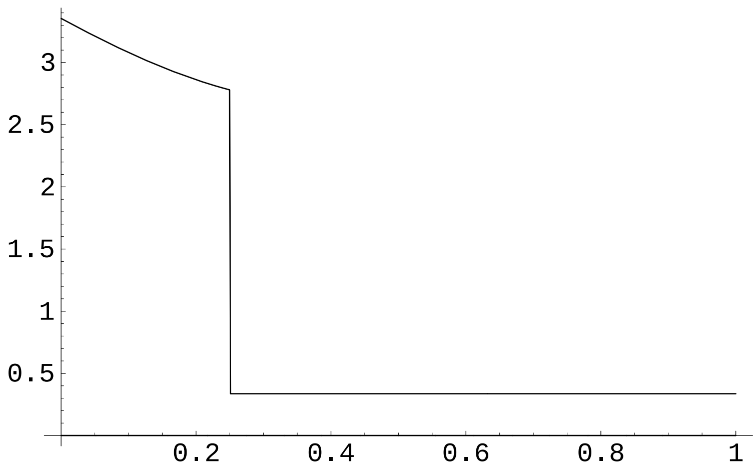
– Graph of $(A_\varphi{}^3 f)(x)$ –



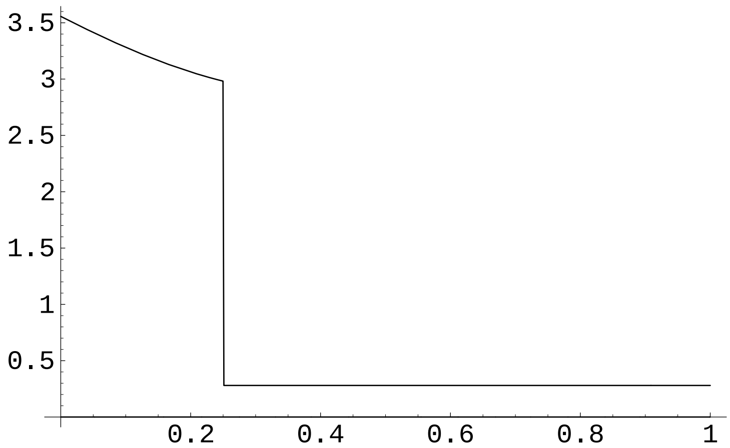
– Graph of $(A_\varphi{}^4 f)(x)$ –



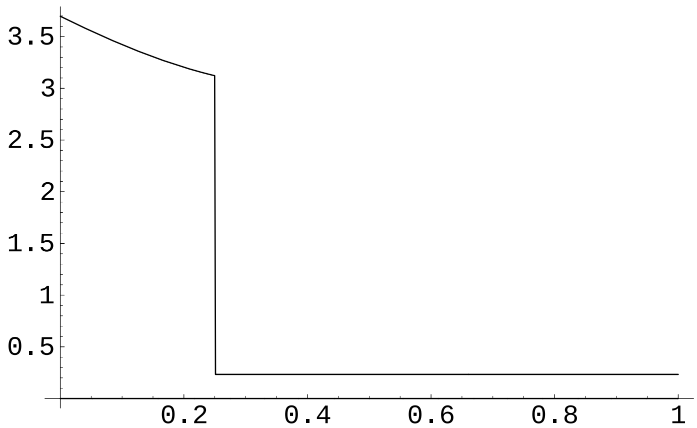
– Graph of $(A_\varphi{}^5 f)(x)$ –



– Graph of $(A_\varphi{}^6 f)(x)$ –



– Graph of $(A_\varphi{}^7 f)(x)$ –



– Graph of $(A_\varphi{}^8 f)(x)$ –

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