Global Journal of Pure and Applied Mathematics
ISSN 0973-1768 Volume 5, Number 3 (2009), pp. 181–187

http://www.ripublication.com/gjpam.htm

2-NSR Lemma and Compact Operators on 2-Normed Space

[1]P. Riyas and [2]K. T. Ravindran

P.G Department and Research Centre in Mathematics, Payyanur College, Payyanur, Kerala, India.
E-mails: [1]riyasmankadavu@gmail.com, [2]ktravindran@rediffmail.com.

Abstract

In this paper we discuss properties of compactness and compact operator on 2-normed space. Also we consider a result which is similar to Riesz Lemma and its applications in 2-normed space. We introduce quotient space from the finite dimensional subspace of a 2-normed space.

AMS Classification: 41A65, 41A15.

Keywords: 2-normed space, locally bounded, compact operator, Riesz Lemma, e-bounded operator.

1. Introduction

The concept of a linear 2-normed space was introduced as a natural 2-metric analogue of that of a normed space. In 1963, Gähler introduced the notion of a 2-metric space, a real valued function of point-triples on a set X, whose abstract properties were suggested by the area function for a triangle determined by a triple in Euclidean space.

We begin with a lemma which was given by Fatemeh Lael and Kourosh Nouruzi[2], by using the norm $\| x+\langle e\rangle \| = \frac{\| \mathrm{x, e} \|}{\| \mathrm{e, e'} \|}$ in the quotient space $\frac{X}{\langle e\rangle}$. Even though this norm seems to be many valued, they have reached the conclusion on the basis of the same norm. Inspired by their finding we have proved the same lemma by using the norm

(1) $\| x+\langle e\rangle \|_Q = \| x,e \| + \sup\left\{ \frac{\| x,e \|}{\| e,e' \|} :\ e' \notin \langle e\rangle \, and \, \| e,e' \| > 1 \right\}$. Thus all the results in paper [2] can be proved by using the specific norm(1) in the quotient space

$\frac{X}{\langle e \rangle}$. Here we prove a lemma which is similar to Riesz lemma in normed space and using this we list the properties of compact operator.

In [4] S. Gähler introduced the following definition of a 2-normed space.

2. Preliminaries

Definition 2.1[4]: Let X be a real linear space of dimension greater than 1. Suppose $\|.,.\|$ is a real valued function on X $\times$X satisfying the following conditions:

A1: $\|x,y\| = 0 \Leftrightarrow$ x and y are linearly dependent.

A2: $\|x,y\| = \|y,x\|$

A3: $\|\alpha x,y\| = |\alpha| \|x,y\|$

A4: $\|x,y+z\| \leq \|x,y\| + \|x,z\|$.

Then $\|.,.\|$ is called a 2-norm on X and the pair $(X, \|.,.\|)$ is called a 2-normed space. Some of the basic properties of 2-norms, that they are non-negative and $\|x,y+\alpha x\| = \|x,y\|$ for all x,y $\in$ X and for all $\alpha \in \Re$.

Definition 2.2[2]: Let X and Y be two 2-normed spaces and T: X $\longrightarrow$ Y be a linear operator. For any e$\in$ X, we say that the operator T is e-bounded if there exist M_e>0 such that $\|T(x), T(e)\| \leq M_e \|x,e\|$ for all x $\in$ X. An e-bounded operator T for every e will be called bounded.

Definition 2.3[4]: A sequence $\{x_n\}$ in a 2-normed space X is said to be convergent if there exists an element x$\in$ X such that $\lim_{n\to\infty} \| x_n - x, y \| = 0$ for all y$\in$ X.

Definition 2.4 [4]: Let X and Y be two 2-normed spaces and T: X $\longrightarrow$ Y be a linear operator. The operator T is said to be sequentially continuous at x$\in$ X if for any sequence $\{x_n\}$ of X converging to x we have $T(x_n) \longrightarrow T(x)$.

Definition 2.5 [2]: The closure of a subset *E* of a 2-normed space X is denoted by $\overline{E}$ and defined by the set of all x$\in$ X such that there is a sequence $\{x_n\}$ of *E* converging to x. We say that *E* is closed if $E = \overline{E}$.

For a 2-normed space, we consider the subsets

$$B_e(a, r) = \{ x : \|x-a,e\| < r \}$$
$$B_e[a, r] = \{ x : \|x-a,e\| \leq r \} \text{ of X.}$$

Definition 2.6 [2]: A subset A of a 2-normed space X is said to be locally bounded if there exist e$\in$ X—{0} and r >0 such that A$\subseteq$ $B_e(0, r)$.

Definition 2.7 [2]: A subset B of a 2-normed space X is said to be compact if every $\{x_n\}$ of B has a convergent subsequence in B.

Definition 2.8 [2]: Let X and Y be two 2-normed spaces. A linear operator T: X $\longrightarrow$ Y is called a compact operator if it maps every locally bounded sequence $\{x_n\}$ of X onto a sequence $\{T(x_n)\}$ in Y which has a convergent subsequence.

Lemma 2.9[2]: Let X and Y be two 2-normed spaces. If T: X $\longrightarrow$ Y is a surjective bounded linear operator then T is sequentially continuous.

Corollary 2.10[2]: Let X and Y be two 2-normed spaces. Then every compact operator T: X $\longrightarrow$ Y is bounded.

3. Main Results

Lemma 3.1: Let X be a 2-normed space. If $B_e[a, r]$ is compact in X for some a,$e \in$ X and r >0 then X is of finite dimension.

Proof: Suppose that $B_e[a, r]$ is compact. The quotient space $\frac{X}{\langle e\rangle}$ is a normed space equipped with the norm;

$$\|x+\langle e\rangle\|_Q = \inf\left\{\frac{\|x,e\|}{\|e,e'\|} : e' \notin \langle e\rangle \, and \, \|e,e'\| \le 1\right\} + \sup\left\{\frac{\|x,e\|}{\|e,e'\|} : e' \notin \langle e\rangle \, and \, \|e,e'\| > 1\right\}$$

$$= \|x,e\| + \sup\left\{\frac{\|x,e\|}{\|e,e'\|} : \; e' \notin \langle e\rangle \, and \, \|e,e'\| > 1\right\}.$$

Define $A_{e'} = \left\{x+\langle e\rangle : \; \|x-a+\langle e\rangle\| \le \frac{r}{\|e,e'\|}\right\}$

and let A = $\bigcap\{A_{e'} : e \text{ and } e' \text{ are linearly independent}\}$. Then A is a closed ball in the normed space $\frac{X}{\langle e\rangle}$. We aim to show that A is a compact set in the normed space $\frac{X}{\langle e\rangle}$. For that let $\{x_n+\langle e\rangle\}$ be any sequence in A.

$$\|x_n+\langle e\rangle-(a+\langle e\rangle)\|_Q = \|x_n-a+\langle e\rangle\|_Q \le \frac{r}{\|e,e'\|}; \; \forall \, e' \notin \langle e\rangle \text{ and } \forall \, n .$$

$$\Rightarrow \qquad \|x_n-a,e\| \le \frac{r}{\|e,e'\|} \; ; \forall e' \in \langle e\rangle \text{ and } \forall n .$$

In particular, $\|x_n-a,e\| \le r$; $\forall n$

$$\Rightarrow \qquad x_n \in B_e[a, r].$$

Hence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ converges to a point x_0.

We have $\|x_{n_k}+\langle e\rangle-(x_0+\langle e\rangle)\|_Q = \|x_{n_k}-x_0+\langle e\rangle\|_Q$

$$= \| x_{n_k} - x_0, e \| + \sup\left\{ \frac{\| x_{n_k} - x_0, e \|}{\| e, e' \|} : \ e' \notin \langle e \rangle \, and \, \| e, e' \| > 1 \right\}$$

$$= \| x_{n_k} - x_0, e \| \left[1 + \sup\left\{ \frac{1}{\| e, e' \|} : \ e' \notin \langle e \rangle \, and \, \| e, e' \| > 1 \right\} \right].$$

$\Rightarrow \lim_{k \to \infty} \| x_{n_k} + \langle e \rangle - (x_0 + \langle e \rangle) \|_Q = 0$.

Hence $\{ x_{n_k} + \langle e \rangle \}$ is a convergent subsequence of $\{ x_n + \langle e \rangle \}$. This implies that A is compact and so $\frac{X}{\langle e \rangle}$ is of finite dimension.

$\Rightarrow$ X is of finite dimension. □

Remark : Here we introduce a result which is similar to Riesz Lemma .

Lemma 3.2(2-NSR LEMMA): Let X be a 2-normed space and let $0 \neq e \in X$. Let r be any number such that 0< r <1. Then there exist some $x_r \in X$ such that $\| x_r, e \| = 1$ and $r < \| x_r, e \| \leq 2$.

Proof: Since $\| x, e \| > 0$, $\forall$ x $\notin \langle e \rangle$,

we have $\| x + \langle e \rangle \|_Q = \| x, e \| + \sup\left\{ \frac{\| x, e \|}{\| e, e' \|} : \ e' \notin \langle e \rangle \, and \, \| e, e' \| \ > 1 \right\}$

$> 0; \ \forall \ x \notin \langle e \rangle$.

Also, as r < 1 $\| x, e \| \leq \| x + \langle e \rangle \|_Q < \frac{\| x + \langle e \rangle \|_Q}{r}$, $\forall \ x \notin \langle e \rangle$.

Put $x_r = \frac{x}{\| x, e \|}$

so that $\| x_r, e \| = 1$ and $\| x_r + \langle e \rangle \| = \left\| \frac{x}{\| x, e \|} + \langle e \rangle \right\| = \frac{1}{\| x, e \|} \| x + \langle e \rangle \| > r$.

$\| x_r + \langle e \rangle \|_Q = \| x_r, e \| + \sup\left\{ \frac{\| x_r, e \|}{\| e, e' \|} : \ e' \notin \langle e \rangle \, and \, \| e, e' \| \ > 1 \right\}$

$= 1 + \sup\left\{ \frac{1}{\| e, e' \|} : \ e' \notin \langle e \rangle \, and \, \| e, e' \| \ > 1 \right\}$

≤ 2.

Thus there exist $x_r \in X$ such that $\| x_r, e \| = 1$ and $r < \| x_r, e \| \leq 2$. □

Remark: Let X be a 2-normed space ad let Y be a finite dimensional subspace of X generated by $\{ e_1, e_2, ... e_n \}$ .Then $\frac{X}{Y}$ is a normed space equipped with the norm

$$\| x + Y \|_Q = \sum_{k=1}^{n} \| x + \langle e_k \rangle \|_Q .$$

Corollary 3.3: Let X be a 2-normed space and let Y be a finite dimensional subspace of X. Let r be any number such that 0 < r < 1.Then there exist some $x_r \in X$ such that $r < \| x_r + Y \| \leq 2$.

Proof: Let $\{e_1, e_2, ...e_n\}$ be a basis for Y. Then for any $x \notin Y$,

$$r < \| \frac{x}{\| x, e_k \|} + \langle e_k \rangle \| \leq 2; \text{ for } k = 1,2,..,n.$$

$\Rightarrow \quad r \| x, e_k \| < \| x + \langle e_k \rangle \| \leq 2 \| x, e_k \|$, for k = 1,2,3,…n and for every $x \notin Y$.

$$\Rightarrow \quad r \sum_{k=1}^{n} \| x, e_k \| < \| x + Y \| \leq 2 \sum_{k=1}^{n} \| x, e_k \|$$

Put $x_r = \dfrac{x}{\sum_{k=1}^{n} \| x, e_k \|}$ so that $r < \| x_r + Y \| \leq 2$. □

Lemma 3.4: Every finite dimensional subspace Y of a 2-normed space X is complete.

Proof: To prove the completeness of Y, we use mathematical induction on the dimension m of Y. Let m = 1.Then Y = {k*e* : k ∈ $\Re$ } with $e \neq 0$. If $\{x_n\}$ is a Cauchy sequence in Y with $x_n = k_n e$, then for every $x \in X$, $\| x_n - x_m, x \| \to 0$

$$\| x_n - x_m, x \| = \| (k_n - k_m) e, x \| = | k_n - k_m | \| e, x \| \ ; \forall x \in X .$$

$$\Rightarrow \quad | k_n - k_m | = \frac{\| x_n - x_m, x \|}{\| x, e \|}, \ \forall x \notin \langle e \rangle .$$

$\Rightarrow \quad \{k_n\}$ is a Cauchy sequence in $\Re$ which is complete.

If $k_n \longrightarrow k$ in $\Re$ then $x_n \longrightarrow ke$ in Y. Thus Y is complete. Now assume that every m - 1 dimensional subspace of X is complete. Let dim Y= m and let $\{x_n\}$ be a Cauchy sequence in Y. Let $\{e_1, e_2, ...e_m\}$ be a basis for Y and let Z = span $\{e_2, ...e_m\}$.Now for each n=1,2,3,… $x_n = k_n e_1 + z_n$ for some $k_n \in \Re$ and $z_n \in Z$.

$$\| x_n - x_m, x \| = \| (k_n - k_m) e_1 + (z_n - z_m), x \| = | k_n - k_m | \| e_1 + \frac{(z_n - z_m)}{(k_n - k_m)}, x \| \ ; \forall x \in X$$

In particular, $\| x_n - x_m, e_2 \| = | k_n - k_m | \| e_1 + \dfrac{(z_n - z_m)}{(k_n - k_m)}, e_2 \| \ ; \forall x \in X$

$$> \frac{| k_n - k_m |}{2} \| e_1 + Z \| .$$

$\Rightarrow \quad \{k_n\}$ is a Cauchy sequence in $\Re$ which is complete. As $z_n = x_n - k_n e_1$, it follows that $\{z_n\}$ is a Cauchy sequence in Z which is complete. If $k_n \longrightarrow k$ in $\Re$ and $z_n \longrightarrow z$ in Z, then $x_n \longrightarrow k e_1 + z$ in Y. Hence Y is complete. □

Theorem 3.5: Let X be a 2-normed space and let T be a surjective compact operator on X and $0 \neq k \in$ R. If $\{x_n\}$ is a locally bounded sequence in X such that

$T(x_n) - kx_n \longrightarrow y$ in X then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to x in X and $T(x) - kx = y$.

Proof: Since T is compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $T(x_{n_k})$ converges to some $z \in X$. Then $k x_{n_k} = \left[kx_{n_k} - T(x_{n_k})\right] + T(x_{n_k})$ converges to $-y + z$ and so $\{x_{n_k}\} \longrightarrow \frac{z-y}{k} = x$. Since T is sequentially continuous[2.9], $T(x_{n_k}) \longrightarrow T(x)$.

$\Rightarrow$ $T(x) - kx = \lim_{k\to\infty}\left[T(x_{n_k}) - kx_{n_k}\right] = z - (-y+z) = y$. □

Theorem 3.6: Let X be a 2-normed space and T: X $\longrightarrow$X. Let $0 \neq k \in$ R and

$0 \neq e \in X$ such that $(T - kI)X \subseteq \langle e\rangle$.Then there is some $x_0 \in X$ such that $\| x_0, e \| = 1$ and for every $y \in \langle e\rangle$, $\| T(x_0) - T(y), e \| > \frac{|k|}{4}$.

Proof: Let $Y = \langle e\rangle$. Then $T(y) = (T(y) - ky) + ky \subseteq Y + Y = Y$, $\forall\ y \in Y$.

$\Rightarrow$ $T(Y) \subseteq Y$. Choose some $x_0 \in X$ such that $\| x_0, e \| = 1$ and $\frac{1}{2} < \| x_0 + \langle e\rangle \|$.

For any $y \in Y$, $\| T(x_0) - T(y), e \| = \| kx_0 - (kx_0 - T(x_0) + T(y)), e \|$

$$= |k| \| x_0 - \frac{1}{k}\left[(kx_0 - T(x_0) + T(y)\right], e \|$$

$$\geq \frac{|k|}{2} \| x_0 - \frac{1}{k}\left[(kx_0 - T(x_0) + T(y)\right] + \langle e\rangle \|_Q$$

$$= \frac{|k|}{2} \| x_0 + \langle e\rangle \|_Q > \frac{|k|}{4}.$$ □

Corollary 3.7: Let X be a 2-normed space and T: X $\longrightarrow$X. Let $0 \neq k \in$ R and let Y be a finite dimensional proper subspace of X such that $(T - kI)X \subseteq Y$.Then there is some $x_0 \in X$ such that for every $x, y \in Y$, $\| T(x_0) - T(y), x \| > \frac{|k|}{4}$.

Proof: Let $\{e_1, e_2, \ldots e_m\}$ be a basis for Y. Then $T(y) = (T(y) - ky) + ky \subseteq Y + Y = Y$, $\forall$ $y \in Y$ and so $T(Y) \subseteq Y$. Choose some $x_0 \in X$ such that $\frac{1}{2} < \| x_0 + Y \|$.

For any $x, y \in Y$,

$\| T(x_0) - T(y), x \| = \| kx_0 - (kx_0 - T(x_0) + T(y)), x \|$

$$= |k| \| x_0 - \frac{1}{k}\left[(kx_0 - T(x_0) + T(y)\right], x \|$$

$$\geq \frac{|k|}{2} \| x_0 - \frac{1}{k}\left[(kx_0 - T(x_0) + T(y)\right] + Y \|$$

$$= \frac{|k|}{2} \| x_0 + Y \| > \frac{|k|}{4}. \qquad \square$$

References

[1] Y.J. Cho, P.C.S. Lin, S.S. Kim and A. Misiak, Theory of 2-Inner product spaces, Nova Science, New York, 2001.

[2] Fatemeh Lael and Kourosh Nouruzi, compact operators defined on 2-normed and 2-Probablistic normed spaces, Mathematical Problems in Engineering Volume 2009 (2009), Article ID 950234,

[3] S. Gähler, Siegfrid 2-metrische Räume und ihre topologische struktur, Math.Nachr.26(1963)115-148.

[4] S. Gähler, Lineare 2- normierte Räume, Math.Nachr.28(1964)1-43.

[5] F. Raymond W and C. Yeol Je, Geometry of Linear 2-normed spaces, Nova science publishers, Inc., Hauppauge, Ny, 2001.