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Absolutely Flat Dimension of Modules

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Abstract

For an $R-$module M, we define absolutely flat dimension which we denote it as A.fd(M). We show that A.fd$(M) \leq n$ if and only if $\text{Tor}_{n+1}^R(M \otimes N, L) = 0$ for every finitely generated $R-$module N and for any $R-$module L. We also prove that if R is noetherian local ring, with k as its quotient field then A.fd$(M) \leq n$ if and only if $\text{Tor}_{n+1}^R(M \otimes N, k) = 0$ for every finitely generated $R-$module N.

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Throughout this article, R denotes a commutative ring with identity and all modules are unitary. For standadrd terminology and notations, the references are [1], [4] and [5].

Absolutely flat modules are introduced and studied in [3]. An $R-$module M is called absolutely flat module if for every $R-$module N, $M \otimes_R N$ is flat $R-$module. In this article for an $R-$module M, we define a numerical constant, absolutely flat dimension, which we denote it as A.fd(M). We prove that A.fd$(M) \leq n$ if and only if $\text{Tor}_{n+1}^R(M \otimes N, L) = 0$ for every finitely generated $R-$module N and for any $R-$module L. We also prove that if R is noetherian local ring, with k as its quotient field then A.fd$(M) \leq n$ if and only if $\text{Tor}_{n+1}^R(M \otimes N, k) = 0$ for every finitely generated $R-$module N.

We need the following definitions and results. The concept of flat dimension is introduced in [2].

Definition 1. Let M be an R-module, the flat dimension of M over R, denoted by fd(M), is equal to the least non-negative integer n, for which there is an exact sequence $0 \to F_n \to F_{n-1} \to \ldots \to F_2 \to F_1 \to F_0 \to M \to 0$. with F_i flat $R-$modules. If no such n exists set fd$(M) = \infty$.

The flat dimension of M, satisfies the following equalities [6]:

$$\begin{aligned} \mathrm{fd}(M) &= \sup\{n \geq 0 \mid \mathrm{Tor}_n^R(M, N) \neq 0, \text{ for some } R\text{−module}N\} \\ &= \inf\{n \geq 0 \mid \mathrm{Tor}_{n+1}^R(M, N) = 0, \text{ for every } R\text{−module}N\}\}. \end{aligned}$$

Definition 2. For a ring R, the weak dimension, abbreviated w.dim(R), is defined as follows:

$$\mathrm{w.dim}(R) = sup\{\mathrm{fd}(M) \mid M \text{ is an } R - \text{module}\}.$$

Proposition 2. (Flat Dimension Theorem) (p.76, [6]) Let M be an R-module. The following are equivalent:

1. $\mathrm{fd}(M) \leq n$
2. $\mathrm{Tor}_{n+1}^R(R/I, M) = 0$, for all finitely generated ideal I.
3. The nth kernel of any flat resolution of M is flat.
4. There exist a flat resolution of M whose nth kernel is flat.
5. There exists a flat resolution $\{F_k, d_k\}$ of M for which $F_k = 0$ when $k > n$.

Proposition 3. Let R be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal and k its residue field,and let M be a finitely generated R-module. Then the following are equivalent:

1. M is free
2. M is flat
3. the mapping of $\mathfrak{m} \otimes_R M$ into $R \otimes_R M$ is injective.
4. $\mathrm{Tor}_1^R(k, M) = 0$.

Proposition 4. If M is an R-module, the following are equivalent:

1. M is flat
2. $\mathrm{Tor}_n^R(M, N) = 0$ for all $n > 0$ and all R-modules N
3. $\mathrm{Tor}_1^R(M, N) = 0$ for all R-modules N.
4. $\mathrm{Tor}_1^R(R/I, M) = 0$ for all finitely generated ideal I in R.

Next we define absolutely flat dimension for an R−module M.

Definition 5. Let M be an R-module. The absolutely flat dimension of M, denoted by A.fd(M) is equal to sup$\{\mathrm{fd}(M \otimes_R N) \mid N$ finitely generatedR-$module\}$.

The absolutely flat dimension of a $R-$module M is related with the flat dimension of M and weak dimension of R as follows:

Proposition 6. For an R-module M, $\mathrm{fd}(M) \leq \mathrm{A.fd}(M) \leq \mathrm{W.dim}(R)$ and the equality holds if R is an absolutely flat ring.

Proof. We note that $\mathrm{fd}(M \otimes_R N) \leq \mathrm{A.fd}(M)$, for every finitely generated R-module N. In particular, for $N = R$, we get $\mathrm{fd}(M) \leq \mathrm{A.fd}(M)$. The other inequality follows from the definition of weak dimension. ■

Next we characterize absolutely flat dimension using Tor functor.

Theorem 6. (Absolutely flat dimension theorem) Let M be an R-module. Then the following are equivalent:

1. $\mathrm{A.fd}(M) \leq n$
2. $\mathrm{Tor}_k^R(M \otimes_R N, L) = 0$, for all finitely generated R-module N, and any R-module L and for all $k \geq n + 1$
3. $\mathrm{Tor}_{n+1}^R(M \otimes_R N, L) = 0$, for all finitely generated R-module N, and any R-module L
4. $\mathrm{Tor}_{n+1}^R(M \otimes_R N, R/I) = 0$, for all finitely generated R-module N, and all finitely generated ideal I.

Proof. (1) $\Rightarrow$ (2) Let $\mathrm{A.fd}(M) \leq n$, Then $\mathrm{fd}(M \otimes_R N) \leq n$ for all finitely generated R-module N. Hence $\mathrm{Tor}_k^R(M \otimes_R N, L) = 0$ for $N, L, k \geq n + 1$.
(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious.
(4)($\Rightarrow$)(1) Assume that $\mathrm{Tor}_{n+1}^R(M \otimes_R N, R/I) = 0$ for all finitely generated R-module N and I, any finitely generated ideal. By flat dimension theorem, (proposition 2), $\mathrm{fd}(M \otimes_R N) \leq n$ for all N. This implies $\mathrm{A.fd}(M) \leq n$. ■

Theorem 7. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M be a finitely generated R-module. Then

$$\mathrm{A.fd}(M) \leq n \Leftrightarrow \mathrm{Tor}_{n+1}^R(M \otimes_R N, k) = 0 \text{ for any finitely generated}$$

$R -$ module N.

Proof. Assume that $\mathrm{A.fd}(M) \leq n$. This implies that $\mathrm{fd}(M \otimes_R N) \leq n$ for any finitely generated R-module N. Therefore $\mathrm{Tor}_{n+1}^R(M \otimes_R N, L) = 0$, for any R-module L. Hence $\mathrm{Tor}_{n+1}^R(M \otimes_R N, k) = 0$.

Conversely, assume that $\mathrm{Tor}_{n+1}^R(M \otimes_R N, k) = 0$ for all finitely generated R-module N. Consider a flat resolution of $M \otimes_R N$,

$$0 \to K_{n-1} \to X_{n-1} \to \cdots \to X_2 \to X_1 \to X_0 \to M \otimes_R N \to 0.$$

That is, the above sequence is exact, $X_0, X_1, X_2, \ldots, X_{n-1}$ are finitely generated flat $R-$modules and K_{n-1} is the $\ker(X_{n-1} \to X_{n-2})$. We split the above long exact sequence into short exact sequences as

$$0 \to K_0 \to X_0 \to M \otimes_R N \to 0$$

$$0 \to K_1 \to X_1 \to K_0 \to 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$0 \to K_{n-1} \to X_{n-1} \to K_{n-2} \to 0,$$

where K_i is the kernel of the homomorphisms $X_i \to K_{i-1}$ for $1 \leq i \leq n$. Since X_i's are flat, by successively applying the functor $\mathrm{Tor}(M \otimes_R N, -)$ on each of these short exact sequence, we have the isomorphisms,

$$\mathrm{Tor}_{n+1}^R(M \otimes_R N, k) \cong \mathrm{Tor}_n^R(K_0, k)$$

$$\mathrm{Tor}_n^R(K_0, k) \cong \mathrm{Tor}_{n-1}^R(K_1, k)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathrm{Tor}_2^R(K_{n-2}, k) \cong \mathrm{Tor}_1^R(K_{n-1}, k).$$

Therefore

$$0 = \mathrm{Tor}_{n+1}^R(M \otimes_R N, k) \cong \mathrm{Tor}_n^R(K_0, k) \cong \mathrm{Tor}_{n-1}^R(K_1, k) ... \cong \mathrm{Tor}_1^R(K_{n-1}, k).$$

Since $M \otimes_R N$ is finitely generated and X_i's are finitely generated and flat, K_{n-1} is finitely generated.

Then by proposition 3, K_{n-1} is flat. Hence we have the flat resolution of $M \otimes_R N$

$$0 \to K_{n-1} \to X_{n-1} \to ... \to X_2 \to X_1 \to X_0 \to M \otimes_R N \to 0.$$

This implies $\mathrm{fd}(M \otimes_R N) \leq n$ where N is any finitely generated R-module. Hence $\mathrm{A.fd}(M) \leq n$. ■

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