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Some Applications of SCS Spaces

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Abstract

We introduce sequentially semi-subcontinuous (inversely sequentially semi-subcontinuous) maps and characterize them as relatively countably semi-compact preserving (relatively countably semi-compact) maps. Characterization of countably semi-compact preserving (countably semi-compact) maps is obtained in terms semi-cluster points of images of semi-converging sequences. We also give characterization of sequentially semi-subcontinuous (inversely sequentially semi-subcontinuous) for presemiclosed (irresolute) map.

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1. Introduction and Preliminaries

In [6], Fuller introduced the concept of subcontinuous functions. Since then, subcontinuity and its variants have been studied by various authors (see for example [7] and [11]). Garg and Goel [7] characterized continuity for sequentially cluster preserving (cluster preserving) maps as well as sequentially subcontinuous maps under suitable restrictions on the domain and range. In [12], sequential subcontinuity (inverse sequential subcontinuity) was characterized for closed (continuous) map. On the other hand, Garg and Shivaraj [8] studied presemiclosed mappings and characterized them in terms of semi-closure (Theorem 1.1 below). In [11], relatively compact preserving (relatively compact) maps were characterized as weakly subcontinuous (weakly inversely subcontinuous) for regular codomain (domain) ([11]; Theorem 2.2).

In this paper, we use SCS spaces (Sarsak; [13]) and introduce sequentially semi-subcontinuous (inversely sequentially semi-subcontinuous) maps and characterize them as relatively countably semi-compact preserving (relatively countably semi-compact) maps [Theorem 2.10 and 2.11 below]. We obtain necessary and sufficient condition for a map to be countably semi-compact preserving (countably semi-compact) by restricting the set in which the semi-cluster point of the image sequence (the sequence) lies for semi-convergent sequence (image sequence), where it is assumed that the domain (co-domain) is semi-Frèchet, SCS [Theorem 2.12 and 2.13 below]. We characterize sequential semi-subcontinuity (inversely sequential semi-subcontinuity) for presemiclosed (irresolute) mappings [Theorem 2.14 and 2.15 below].

A subset A in a topological space X is said to be semi-open [9] if and only if $A \subset cl(Int(A))$, or equivalently, if there exists an open subset U of X such that $U \subset A \subset cl(U)$. A is called semi-closed if X-A is semi-open. The semi-closure $scl(A)$ of a subset A of a space X is the intersection of all semi-closed subsets of X that contain A, or equivalently, the smallest semi-closed subset of X that contains A. A space X is called semi-compact [5] (countably semi-compact [4]) if any cover (countable cover) of X by semi-open sets has a finite subcover. A net $\{x_\alpha\}$ semi-converges [5] (semi-clusters [3]) at x if and only if $\{x_\alpha\}$ is eventually (frequently) in every semi-open set containing x. A function $f : X \to Y$ is said to be irresolute [1] if for any semi-open set S of Y, $f^{-1}(S)$ is semi-open in X. A function $f : X \to Y$ is presemiclosed [8] if and only if image set $f(A)$ is semi-closed for each semi-closed subset A of X. A space is said to be SCS [13] if any subset of X which is semi-compact is semi-closed. A space is semi-T_1 [10] if for any two points x, y of X there exists a semi-open set containing x but not y and a semi- open set containing y but not x. Clearly semi-T_1 is strictly weaker than T_1 and every SCS space is semi-T_1. A topological space X is said to be extremely disconnected [2] if the closure of every open set is also an open set, or equivalently, collection of all semi-open sets is a topology on X.

Notation: Throughout this paper, X and Y will denote extremely disconnected topological spaces. For a subset A of a space X, scl(A), cl(A) and int(A) will denote the semi-closure of A, closure of A and interior of A respectively.

We will also make use of following results:

Theorem 1.1. [8, Theorem 3.3] A mapping $f : X \to Y$ is presemiclosed if and only if $scl(f(A)) \subset f(scl(A))$ for every subset A of X.

Theorem 1.2. [1, Theorem 1.6] A mapping $f : X \to Y$ is irresolute if and only if $scl(f^{-1}(A)) \subset f^{-1}(scl(A))$ for every subset A of Y.

2. Results

Definition 2.1. A subset A of X is called **relatively countably semi-compact** if scl(A) is countably semi-compact.

Remark 2.2. Since semi-closed subset of countably semi-compact space is countably semi-compact, therefore every subset of countably semi-compact is relatively countably semi-compact.

The following lemma characterizes countably semi-compactness in terms of sequences and its proof is just analogous to countably compact space.

Lemma 2.3. In a space X, a set A is countably semi-compact if and only if every sequence in A has a semi-cluster point in A.

Lemma 2.4. If A is relatively countably semi-compact in a space X then every sequence in A has a semi-cluster point in X.

Proof. Obvious from Lemma 2.3. ■

Definition 2.5. A map $f : X \to Y$ is called

(a) **Countably semi-compact preserving (countably semi-compact)** if the image (inverse image) of a countably semi-compact subset of X (Y) is countably semi-compact in Y(X).

(b) **Relatively countably semi-compact preserving (Relatively countably semi-compact)** if the image (inverse image) of a relatively countably semi-compact subset of X (Y) is relatively countably semi-compact in Y (X).

Definition 2.6. A space X is said to be **semi-Frèchet** if for each subset A of X, $x \in scl(A)$ implies there exists a sequence $\{x_n\}$ in A semi-converging to x.

Definition 2.7. A map $f : X \to Y$ will be called **sequentially semi-subcontinuous (inversely sequentially semi-subcontinuous)** if whenever the sequence $\{x_n\}$ (the sequence $\{f(x_n)\}$ has a semi-cluster point in X (Y), the sequence $\{f(x_n)\}$ ($\{x_n\}$) has a semi-cluster point in Y (X).

The proof of following Lemma is similar to that of Lemma 2.1 of [12].

Lemma 2.8. In a semi-Frèchet, semi-T_1 space, if a sequence $\{x_n\}$ has a semi-cluster point x, then $\{x_n\}$ has a subsequence semi-converging to x.

For the next theorem we will make use of following result, the proof is straightforward and is omitted:

Lemma 2.9. In any space X, if a sequence $\{x_n\}$ semi-converging to x then $\cup_n\{x_n, x\}$ is semi-compact subset of X.

The following Theorem gives sufficient conditions for a map $f : X \to Y$, where X is semi-Frèchet, SCS to be sequentially semi-subcontinuous.

Theorem 2.10. If a map $f : X \rightarrow Y$, where X is semi-Frèchet, SCS, is relatively countably semi-compact preserving, then f is sequentially semi-subcontinuous.

Proof. If a sequence $\{x_n\}$ has a semi-cluster point x in X, then by Lemma 2.8, there exists a subsequence $\{x_n*\}$ of $\{x_n\}$ such that $\{x_n*\}$ semi-converging to x in X and so $\cup_n\{x_n*, x*\}$ is semi-compact by Lemma 2.9 and therefore, semi-closed since X is SCS. Since every semi-compact set is countably semi-compact, $\cup_n\{x_n*, x*\}$ is countably semi-compact and therefore, relatively countably semi-compact. Then f is relatively countably semi-compact preserving implies that the sequence $\{f(x_n*)\}$ and therefore, the sequence $\{f(x_n)\}$ by Lemma 2.4. Hence f is sequentially semi-subcontinuous. ■

The following Theorem gives sufficient conditions for a map $f : X \rightarrow Y$ to be inversely sequentially semi-subcontinuous, where X (Y) is semi-Frèchet, SCS. Its proof is analogous to above theorem.

Theorem 2.11. If a map $f : X \rightarrow Y$, where Y is semi-Frèchet, SCS, is relatively countably semi-compact, then f is inversely sequentially semi-subcontinuous.

The following Theorem 2.12 characterizes countably semi-compact preserving map of a semi-Frèchet, SCS space in terms of semi-cluster points of images of semi-converging sequences.

Theorem 2.12. Let $f : X \rightarrow Y$ be any map, where X is semi-Frèchet, SCS. Then f is countably semi-compact preserving if and only if for any sequence $\{x_n\}$ in X , $\{x_n\}$ semi-converging to x in X implies that the sequence $\{f(x_n)\}$ has a semi-cluster point in the subset $S = \cup_n\{f(x_n), f(x)\}$ of Y.

Proof. For arbitrary spaces X and Y, if f is countably semi-compact preserving and $\{x_n\}$ semi-converging to x in X, then by Lemma 2.9, $\cup_n\{x_n, x\}$ is semi-compact in X, which implies that the set S is a countably semi-compact subset of Y. Therefore, the sequence $\{f(x_n)\}$ has a semi-cluster point in S. Conversely, let X semi-Frèchet, SCS and assume that the given condition holds. Let K be a countably semi-compact subset of X. Let $\{f(x_n)\}$ be any sequence in f (K). Then $\{x_n\}$ can be chosen to be a sequence in K and so has a semi-cluster point x in K by Lemma 2.3. Since X is semi-Frèchet, SCS, by Lemma 2.8 there exists a subsequence $\{x_n*\}$ of $\{x_n\}$ such that $\{x_n*\}$ semi-converging to x in X and so by hypothesis the sequence $\{f(x_n*)\}$ and therefore, the sequence $\{f(x_n)\}$ has a semi-cluster point in $S \subset f(K)$. Therefore, f(K) is countably semi-compact. Hence f is countably semi-compact preserving. ■

Proof of the following theorem is analogous to above theorem.

Theorem 2.13. Let $f : X \rightarrow Y$ be any map, where Y is semi-Frèchet, SCS. Then f is countably semi-compact if and only if for any sequence $\{f(x_n)\}$ in Y, $\{f(x_n)\}$ semi-converging to y in Y implies that the sequence $\{x_n\}$ has a semi-cluster point in the subset $S = \cup_n\{f^{-1}(x_n), f^{-1}(y)\}$ of X.

The following Theorem which characterizes sequential semi-subcontinuity for presemiclosed maps is a corollary to the above Theorem 2.12.

Theorem 2.14. Let $f : X \to Y$ be a presemiclosed map, where X is semi-Frèchet, SCS. Then f is countably semi-compact preserving if and only if f is sequentially semi-subcontinuous.

Proof. If any map f is countably semi-compact preserving and a sequence $\{x_n\}$ has a semi-cluster point x in X, by Lemma 2.8, the sequence $\{x_n\}$ has a subsequence $\{x_n*\}$ such that $\{x_n*\}$ semi-converging to x in X. Then by Theorem 2.12, the subsequence $\{f(x_n)*\}$ and so the sequence $\{f(x_n)\}$ has a semi-cluster point in Y. Hence f is sequentially semi-subcontinuous. Conversely, if a presemiclosed map is sequentially semi-subcontinuous and $\{x_n\}$ semi-converging to x in X, then the sequence $\{f(x_n)\}$ has a semi-cluster point y*. Since f is presemiclosed and X is SCS, the semi-cluster point y* must belong to the semi-closed set $\cup_n\{f(x_n), f(x)\}$. Hence by Theorem 2.12, f is countably semi-compact preserving. ■

The following Theorem characterizes inversely sequentially semi-subcontinuity for irresolute maps. Proof is analogous to above theorem.

Theorem 2.15. Let $f : X \to Y$ be a irresolute map, where Y is semi-Frèchet, SCS. Then f is countably semi-compact if and only if f is inversely sequentially semi-subcontinuous.

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