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Controllability of Impulsive Neutral Functional Integrodifferential Inclusions with an Infinite Delay

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Abstract

In this present work, we prove the sufficient condition for the controllability of impulsive neutral functional integrodifferential inclusions with an infinite delay in Banach spaces. The results are obtained by using fixed point theorem for condensing maps due to Martelli.

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1. Introduction

In the past three decades, the mathematical explanations of many hybrid dynamical systems have an impulsive attitude due to sudden changes at certain instants during the evolution process. Recent development in the theory of impulsive differential equations and inclusions has been object interest because of its wide applications in medical domains, industry, information science, system and control, communication security and space techniques see for instance ([16, 29]). These processes tend to be more suitably modeled by impulsive systems which allow for discontinuities in the evolution of the state. For more details about this theory and its applications, we refer to the monographs of Bainov and Simeonov [1], Lakshmikantham et al. [18] and Samoilenko and Perestyuk [26] and the papers of [11, 12, 21].

On the other hand, controllability is one of the fundamental and important concepts in mathematical control theory. This is the qualitative and quantitative property of dynamical control systems and is of particular importance in control theory. For example, Balachandran et al. [25], Benchohra et al. [4, 5, 6], Gunasekar et al. [23], Chang et al. [8, 9], Vijayakumar et al. [27] and Jose et al. [17] discussed the controllability of functional differential and integrodifferential inclusions in Banach spaces with the help of some fixed-point theorems. Specially, it should be pointed out that Benchohra and Ntouyas [7] investigate the controllability for neutral functional differential integrodifferential inclusions with the aid of a fixed-point theorem for condensing maps due to Martelli. Since many systems arising from realistic models heavily depend on histories (i.e., there is the effect of infinite delay on state equations) [28], there is a real need to discuss partial functional differential systems with infinite delay.

Recently, in many areas of science there has been an increasing interest in the investigation of functional differential equations incorporating memory or aftereffect, i.e., there is the effect of infinite delay on state equations. Therefore, there is a real need to discuss functional differential systems with infinite delay. And the development of the theory of functional differential equations with infinite delays depends on a choice of a phase space. In fact, various phase spaces have been considered and each different phase space has required a separate development of the theory (Hino et al. [14]). The common space is the phase space $\mathcal{B}$ proposed by Hale and Kato [13], which is widely applied in functional differential equations with infinite delay and references therein. However, in this paper, we introduce an abstract phase space $\mathcal{B}_h$ which has been adopted by [2, 25, 8].

Very recently, by using the same fixed point theorem, Liu [20] proved the controllability of the first order impulsive partial neutral functional differential inclusions with infinite delay:

$$\begin{aligned} \frac{d}{dt}[x(t) - g(t, x_t)] &\in Ax(t) + F(t, x_t) + (Bu)(t), \ t \in J = [0, b], \ t \neq t_k, \\ \Delta x|_{t=t_k} &= I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m, \\ x_0 &= \phi \in \mathcal{B}_h. \end{aligned}$$

To the best of our knowledge, there is no work reported on a controllability of impulsive partial neutral functional integrodifferential inclusions with an infinite delay by using abstract phase space $\mathcal{B}_h$ and using fixed point theorem for condensing maps due to Martelli. To close the gap, motivated by the above works, the purpose of this paper is to study the controllability of impulsive neutral functional integrodifferential inclusions with an infinite delay

$$\frac{d}{dt}[u(t) - p(t, u_t)] \in A[u(t) - p(t, u_t)] + F\Big(t, u_t, \int_0^t a(t, s, u_s)ds\Big) + Gx(t),$$

$$t \in J = [0, b], \ t \neq t_k, \quad k = 1, 2, \ldots, m, \tag{1.1}$$

$$\Delta u|_{t=t_k} = I_k(u(t_k^-)), \qquad k = 1, 2, \ldots, m, \tag{1.2}$$

$$u_0 = \varphi \in \mathcal{B}_h, \tag{1.3}$$

where the state variable $u(\cdot)$ takes values in Banach space X with the norm $|\cdot|$; A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X, $G : X \to X$ is a bounded linear operator from a Banach space U of admissible control function into X and the control function $x \in L^2(J, U)$. The function $F : J \times \mathcal{B}_h \times X \to 2^X$ is a bounded, closed, convex-valued map, $p : J \times \mathcal{B}_h \to X, \ a : J \times J \times \mathcal{B}_h \to X, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b, \quad I_k : X \to X, k = 1, 2, \ldots, m$ and $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^-)$ and $u(t_k^+)$ represent the left and right limits of $u(t)$ at $t = t_k$, respectively. The histories $u_t : (-\infty, 0] \to X, u_t(s) = u(t + s), s \leq 0$, belong to an abstract phase space $\mathcal{B}_h$ which is defined in Section 2.

2. Preliminaries

At first, we present the abstract phase space $\mathcal{B}_h$, which has been used in [8, 2, 25]. Assume that $h : (-\infty, 0] \to (0, +\infty)$ is a continuous function with $\ell = \displaystyle\int_{-\infty}^0 h(t)dt < +\infty$. For any $e > 0$, we define

$$\mathcal{B} = \{\psi : [-e, 0] \to X \text{ such that } \psi(t) \text{ is bounded and measurable}\},$$

and equip the space $\mathcal{B}$ with the norm

$$\|\psi\|_{[-e,0]} = \sup_{s\in[-e,0]} |\psi(s)|, \quad \forall\, \psi \in \mathcal{B}.$$

Let us define

$$\mathcal{B}_h = \{\psi : (-\infty, 0] \to X \text{ such that for any } c > 0, \psi|_{[-c,0]} \in \mathcal{B} \text{ and } \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds < +\infty\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\psi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds, \quad \forall\, \psi \in \mathcal{B}_h,$$

then it is clear that $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.

Now we consider the space

$$\mathcal{B}'_h = \{u : (-\infty, b] \to X \text{ such that } u_k \in C(J_k, X) \text{ and there exist } u(t_k^+) \text{ and } u(t_k^-) \text{ with } u(t_k) = u(t_k^-),\ u_0 = \varphi \in \mathcal{B}_h,\ k = 0, 1, .., m\},$$

where u_k is the restriction of u to $J_k = (t_k, t_{k+1}], k = 0, 1, .., m$. Set $\|\cdot\|_b$ be a seminorm in $\mathcal{B}'_h$ defined by

$$\|u\|_b = \|\varphi\|_{\mathcal{B}_h} + \sup\{|u(s)| : s \in [0, b]\}, u \in \mathcal{B}'_h.$$

Next, we introduce definitions, notations and preliminary facts from multivalued analysis which are used throughout this paper.

The notation $C(J, X)$ is the Banach space of continuous functions from J into X with the norm $\|u\|_\infty = \sup_{t\in J} |u(t)|$ for $u \in C(J, X)$. $B(X)$ denotes the Banach space of bounded linear operator from X into X. A measurable function $u : J \to X$ is Bochner integrable if and only if $|u|$ is Lebesgue integrable (For properties of the Bochner integrable see Yosida [30]. $L^1(J, X)$ denotes the Banach space of continuous functions $u : J \to X$ which are Bochner integrable norm by $\|u\|_{L^1} = \int_0^b |u(t)|dt$ for all $u \in L^1(J, X)$.

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $\mathcal{F} : X \to 2^X$ is convex (closed) valued, if $\mathcal{F}(u)$ is convex (closed) for all $u \in X$. $\mathcal{F}$ is bounded on bounded set if $\mathcal{F}(B) = \bigcup_{u\in B} \mathcal{F}(u)$ is bounded in X, for any bounded set B of X (i.e., $\sup_{u\in B} \sup\{\|y\| \in \mathcal{F}(u)\} < \infty$). $\mathcal{F}$ is called upper semicontinuous (u.s.c.) on X if for each $u_* \in X$, the set $\mathcal{F}(u_*)$ is nonempty, closed subset of X, and if for each open set B of X containing

$\mathcal{F}(u_*)$, there exists an open neighbourhood V of u_* such that $\mathcal{F}(V) \subset B$. $\mathcal{F}$ is said to be completely continuous if $\mathcal{F}(B)$ is relatively compact, for every bounded subset $B \subset X$.

If the multivalued map $\mathcal{F}$ is completely continuous with nonempty compact values, then $\mathcal{F}$ is u.s.c. if and only if $\mathcal{F}$ has a closed graph (i.e., $u_n \to u_*$, $y_n \to y_*$, $y_n \in \mathcal{F}u_n$ imply $y_* \in \mathcal{F}u_*$). $\mathcal{F}$ has a fixed point if there is $u \in X$ such that $u \in \mathcal{F}u$.

Let $BCC(X)$ denote the set of all nonempty, bounded, closed and convex subsets of X. A multivalued map $\mathcal{F} : J \to BCC(X)$ is said to be measurable if for each $u \in X$ the function $G_1 : J \to \mathbb{R}$ defined by

$$G_1(t) = d(u, \mathcal{F}(t)) = \inf\{|u - y| : y \in \mathcal{F}(t)\}$$

belongs to $L^1(J, \mathbb{R})$. For more details on multivalued maps see the books of Deimling [10] and Hu et al. [15].

An upper semicontinuous map $H : X \to X$ is said to be condensing [3] if for any subset $B \subset X$ with $\alpha(B) \neq 0$, we have $\alpha(H(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompactness [3]. It is easy to see that a completely continuous multivalued map is a condensing map.

For our approach, we need the following fixed point theorem.

Theorem 2.1. **[Martelli [22]]** Let X be a Banach space and $\Phi : X \to BCC(X)$ a condensing map. If the set

$$\Lambda = \{u \in X : \lambda u \in \Phi u,\ for some\ \lambda > 1\}$$

is bounded then Φ has a fixed point.

3. Controllability Results

In this section, we prove the controllability of the systems (1.1)-(1.3). First, we give the mild solution for the problem (1.1)-(1.3).

Definition 3.1. A function $u : (-\infty, b] \to X$ is called a mild solution of problem (1.1)-(1.3) if the following holds: $u_0 = \varphi \in \mathcal{B}_h$ on $(-\infty, 0]$; $\Delta u|_{t=t_k} = I_k(u(t_k^-)),\ k = 1, 2, \ldots, m$, the restriction of $u(\cdot)$ to the interval $[0, b) - \{t_1, t_2, \ldots, t_m\}$ is continuous, and for each $s \in [0, t)$, the impulsive integral equation

$$\begin{aligned} u(t) = T(t)[\varphi(0) - p(0, \varphi)] + p(t, u_t) + \int_0^t T(t-s)f(s)ds \\ + \int_0^t T(t-s)(Gx)(s)ds \\ + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k^-)), \quad t \in J \end{aligned} \tag{3.1}$$

is satisfied, where

$$f \in S_{F,u} = \left\{ f \in L^1(J, X) : f(t) \in F\Big(t, u_t, \int_0^t a(t, s, u_s)ds\Big),\ \text{for a.e. } t \in J \right\}.$$

Definition 3.2. The system (1.1)-(1.3) is said to be controllable on the interval J if for every $u_0 \in \mathcal{B}_h, u_1 \in X$, there exists a control $x \in L^2(J, U)$ such that the mild solution $u(t)$ of (1.1)-(1.3) satisfies $u(b) = u_1$.

For the study of the problem (1.1)-(1.3), we need the following hypotheses:

(H1) A is the infinitesimal generator of a compact semigroup of bounded linear operator $T(t)$ in X and there exists positive constant M such that $|T(t)| \leq M$.

(H2) G is a continuous operator from U to X and the linear operator $W : L^2(J, U) \to X$, defined by

$$Wx = \int_0^b T(b-s)Gx(s)ds,$$

has a bounded invertible operator W^{-1}, which takes values in $L^2(J, U)/ker W$ and there exist positive constants M_1, M_2 such that $\|G\| \leq M_1$ and $\|W^{-1}\| \leq M_2$ (see [24]).

(H3) $I_k \in C(X, X)$ and there exist constants d_k such that $\|I_k(u)\| \leq d_k,\ k = 1, 2, \ldots, m$ for each $u \in X$.

(H4) (i) The function $p : J \times \mathcal{B}_h \to X$ is continuous and there exists a constant $L > 0$ such that the function p satisfies the Lipschitz condition:

$$|p(t_1, u_1) - p(t_2, u_2)| \leq L(|t_1 - t_2| + \|u_1 - u_2\|_{\mathcal{B}_h}), \quad t_1, t_2 \in J,\ u_1, u_2 \in \mathcal{B}_h.$$

(ii) There exist constants L_1, L_2 such that $\ell L_1 < 1$ and

$$|p(t, u)| \leq L_1\|u\|_{\mathcal{B}_h} + L_2, \quad t \in J,\ u \in \mathcal{B}_h,$$

where $\ell = \int_{-\infty}^0 h(s)ds < +\infty$.

(H5) (i) $F : J \times \mathcal{B}_h \times X \to BCC(X); (t, u, y) \to F(t, u, y)$ is measurable with respect to t for each $u \in \mathcal{B}_h,\ y \in X$, u.s.c. with respect to u, y for each $t \in J$, and for each fixed $u \in \mathcal{B}_h,\ y \in X$, the set

$$S_{F,u} = \left\{ f \in L^1(J, X) : f(t) \in F\Big(t, u_t, \int_0^t a(t, s, u_s)ds\Big), \text{ for a.e. } t \in J \right\}$$

is nonempty.

(ii) There exists an integrable function $m : J \to [0, \infty)$ such that

$$\begin{aligned}\|F\Big(t, u, y\Big)\| &= \sup\Big\{|f| : f \in F\Big(t, u, y\Big)\Big\} \\ &\leq m(t)\Omega(\|u\|_{\mathcal{B}_h} + \|y\|),\ t \in J,\ u \in \mathcal{B}_h, \\ &\quad y \in X,\end{aligned}$$

where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(iii) For each $(t, s) \in J \times J$, the function $a(t, s, \cdot) : \mathcal{B}_h \to X$ is continuous and for each $u \in \mathcal{B}_h$, the function $a(\cdot, \cdot, u) : J \times J \to X$ is strongly measurable. There exist an integrable function $p : J \to [0, \infty)$ and a constant $\gamma > 0$, such that

$$\|a(t, s, u)\| \leq \gamma p(s)\Theta(\|u\|_{\mathcal{B}_h})$$

where $\Theta : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function. Assume that the finite bound of $\int_0^t \gamma p(s)ds$ is L_0.

(H6) The following inequality holds:

$$\int_0^b \widetilde{m}(s)ds < \int_{h_1}^{\infty} \frac{ds}{\Omega(s) + \Theta(s)},$$

where

$$h_1 = \frac{\|\varphi\|_{\mathcal{B}_h} + \ell K_1}{1 - \ell L_1},$$
$$h_2 = \frac{\ell M_1}{1 - \ell L_1}, \widetilde{m}(t) = \max\{h_2, \gamma p(t)\}, t \in J,$$

$$K_1 = M[|\varphi(0)| + L_1\|\varphi\|_{\mathcal{B}_h} + L_2] + L_2 + M\sum_{k=1}^{m} d_k + MbN$$

and

$$N = M_1M_2\Big[|u_1| + M[|\varphi(0)| + L_1\|\varphi\|_{\mathcal{B}_h} + L_2] + L_1\|u_b\|_{\mathcal{B}_h} + L_2 + M\int_0^b \Omega\big[\|u_s\|_{\mathcal{B}_h} + L_0\Theta(\|u_\tau\|_{\mathcal{B}_h})\big]m(s)ds + M\sum_{k=1}^{m} d_k\Big].$$

Remark 3.3.

(i) If $\dim X < \infty$, then for each $u \in \mathcal{B}_h$, $S_{F,u} \neq \emptyset$ (See [19]).

(ii) $S_{F,u}$ is nonempty if and only if the function $Y : J \to \mathbb{R}$ defined by $Y(t) = \inf\{|f| : f \in F(t, u, y)\}$ belongs to $L^1(J, \mathbb{R})$.

Lemma 3.4. (Lasota and Opial [19]) Let J be a compact real interval and X be a Banach space. Let F be a multi-valued map satisfying (H5)(i) and let Γ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator $\Gamma \circ S_F : C(J, X) \to BCC(C(J, X))$, $\quad u \mapsto (\Gamma \circ S_F)(u) := \Gamma(S_{F,u})$ is a closed graph operator in $C(J, X) \times C(J, X)$.

At first, using hypothesis (H2) for an arbitrary function $u(\cdot)$, define the control

$$x(t) = W^{-1}\Big\{u_1 - T(b)[\varphi(0) - p(0,\varphi] - p(b,u_b) - \int_0^b T(b-s)f(s)ds \\ - \sum_{0<t_k<b} T(b-t_k)I_k(u(t_k^-))\Big\}(t),$$

$$f \in S_{F,u} = \left\{ f \in L^1(J,X) : f(t) \in F\Big(t, u_t, \int_0^t a(t,s,u_s)ds\Big), \text{ for a.e. } t \in J \right\}.$$

Lemma 3.5. [20] Assume $u \in \mathcal{B}_h'$, then for $t \in J$, $u_t \in \mathcal{B}_h$. Moreover,

$$\ell|u(t)| \leq \|u_t\|_{\mathcal{B}_h} \leq \|u_0\|_{\mathcal{B}_h} + \ell \sup_{s\in[0,t]} |u(s)|,$$

where $\ell = \int_{-\infty}^0 h(t)dt < +\infty$.

Consider the multivalued map $\Phi : \mathcal{B}_h' \to 2^{\mathcal{B}_h'}$ defined by Φu the set of $\rho \in \mathcal{B}_h'$ such that

$$\rho(t) = \begin{cases} \varphi(t), \ \text{if } t \in (-\infty, 0] \\ T(t)[\varphi(0) - p(0,\varphi)] + p(t,u_t) + \int_0^t T(t-s)f(s)ds \\ + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k^-)) \\ + \int_0^t T(t-\eta)BW^{-1}\Big\{u_1 - T(b)[\varphi(0) - p(0,\varphi] - p(b,u_b) \\ - \int_0^b T(b-s)f(s)ds \\ - \sum_{0<t_k<b} T(b-t_k)I_k(u(t_k^-))\Big\}(\eta)d\eta, \quad t \in J, \end{cases}$$

where $f \in S_{F,u}$.

We shall show that the operators Φ has fixed points, which are then a solution of system (1.1)-(1.3). Clearly, $u_1 \in (\Phi u)(b)$.

For $\varphi \in \mathcal{B}_h$, we define $\widetilde{\varphi}$ by

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0], \\ T(t)\varphi(0), & t \in J, \end{cases} \qquad \text{then } \tilde{\varphi} \in \mathcal{B}_h'.$$

Let $u(t) = y(t) + \tilde{\varphi}(t), -\infty < t \le b$. It is easy to see that u satisfies (3.1) if and only if y satisfies $y_0 = 0$ and

$$y(t) = -T(t)p(0,\varphi) + p(t, y_t + \tilde{\varphi}_t) + \int_0^t T(t-s)f(s)ds + \sum_{0<t_k<t} T(t-t_k)\times$$
$$I_k(y(t_k^-) + \tilde{\varphi}(t_k^-)) + \int_0^t T(t-\eta)BW^{-1}\Big\{u_1 - T(b)[\varphi(0) - p(0,\varphi] - p(b, y_b + \tilde{\varphi}_b)$$
$$- \int_0^b T(b-s)f(s)ds - \sum_{0<t_k<b} T(b-t_k)I_k(y(t_k^-) + \tilde{\varphi}(t_k^-))\Big\}(\eta)\, d\eta, \quad t \in J.$$

Let $\mathcal{B}_h'' = \{y \in \mathcal{B}_h' : y_0 = 0 \in \mathcal{B}_h\}$. For any $y \in \mathcal{B}_h''$,

$$\|y\|_b = \|y_0\|_{\mathcal{B}_h} + \sup\{|y(s)| : 0 \le s \le b\} = \sup\{|y(s)| : 0 \le s \le b\},$$

thus $(\mathcal{B}_h'', \|\cdot\|_b)$ is a Banach space. Set $B_r = \{y \in \mathcal{B}_h'' : \|y\|_b \le r\}$ for some $r \ge 0$, then $B_r \subseteq \mathcal{B}_h''$ is uniformly bounded, and for $y \in B_r$, from Lemma 3.2, we have

$$\begin{aligned}\|y_t + \tilde{\varphi}_t\|_{\mathcal{B}_h} &\le \|y_t\|_{\mathcal{B}_h} + \|\tilde{\varphi}_t\|_{\mathcal{B}_h}\\ &\le \ell \sup_{s\in[0,t]} |y(s)| + \|y_0\|_{\mathcal{B}_h} + \ell \sup_{s\in[0,t]} |\tilde{\varphi}(s)| + \|\tilde{\varphi}_0\|_{\mathcal{B}_h}\\ &\le \ell(r + M|\varphi(0)|) + \|\varphi\|_{\mathcal{B}_h} = r'.\end{aligned} \tag{3.2}$$

Define the multivalued map $\Phi_1 : \mathcal{B}_h'' \to 2^{\mathcal{B}_h''}$ defined by $\Phi_1 y$ the set of $\bar{\rho} \in \mathcal{B}_h''$ such that

$$\bar{\rho}(t) = \begin{cases} 0, \text{ if } t \in (-\infty, 0] \\ -T(t)p(0,\varphi) + p(t, y_t + \tilde{\varphi}_t) + \int_0^t T(t-s)f(s)ds + \sum_{0<t_k<t} T(t-t_k)\times \\ I_k(y(t_k^-) + \tilde{\varphi}(t_k^-)) + \int_0^t T(t-\eta)BW^{-1}\Big\{u_1 - T(b)[\varphi(0) \\ \quad -p(0,\varphi] - p(b, y_b + \tilde{\varphi}_b) \\ - \int_0^b T(b-s)f(s)ds - \sum_{0<t_k<b} T(b-t_k)I_k(y(t_k^-) + \tilde{\varphi}(t_k^-))\Big\}(\eta)d\eta, \quad t \in J. \end{cases}$$

where $f \in S_{F,u}$.

Lemma 3.6. If the hypotheses (H1)-(H5) are satisfied, then $\Phi_1 : \mathcal{B}_h'' \to 2^{\mathcal{B}_h''}$ is a completely continuous multivalued, u.s.c. with a convex closed value.

Proof. We divide the proof into several steps.

Step 1: $\Phi_1 y$ is convex for each $y \in \mathcal{B}_h''$. In fact, if $\bar{\rho}_1, \bar{\rho}_2$ belong to $\Phi_1 y$, then there exist $f_1, f_2 \in S_{F,y}$ such that for each $t \in J$, we have

$$\begin{aligned}\bar{\rho}_i(t) &= -T(t)p(0,\varphi) + p(t, y_t + \tilde{\varphi}_t) + \int_0^t T(t-s)f_i(s)ds \\ &\quad + \sum_{0<t_k<t} T(t-t_k) \times I_k(y(t_k^-) + \tilde{\varphi}(t_k^-)) \\ &\quad + \int_0^t T(t-\eta)BW^{-1}\Big\{u_1 - T(b)[\varphi(0) - p(0,\varphi] - p(b, y_b + \tilde{\varphi}_b) \\ &\quad - \int_0^b T(b-s)f_i(s)ds - \sum_{0<t_k<b} T(b-t_k)I_k(y(t_k^-) + \tilde{\varphi}(t_k^-))\Big\}(\eta)d\eta, \quad i = 1, 2.\end{aligned}$$

Let $\lambda \in [0, 1]$, By operator G and W^{-1}(since $W$is linear)are linear, we have

$$\begin{aligned}&(\lambda\bar{\rho}_1 + (1-\lambda)\bar{\rho}_2)(t) \\ &= -T(t)p(0,\varphi) + p(t, y_t + \tilde{\varphi}_t) + \int_0^t T(t-s)[\lambda f_1(s) + (1-\lambda)f_2(s)]ds \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-) + \tilde{\varphi}(t_k^-)) \\ &+ \int_0^t T(t-\eta)BW^{-1}\Big\{u_1 - T(b)[\varphi(0) - p(0,\varphi] \\ &- p(b, y_b + \tilde{\varphi}_b) - \int_0^b T(b-s)[\lambda f_1(s) + (1-\lambda)f_2(s)]ds \\ &- \sum_{0<t_k<b} T(b-t_k)I_k(y(t_k^-) + \tilde{\varphi}(t_k^-))\Big\}\ (\eta)\ d\eta.\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have $\lambda\bar{\rho}_1 + (1-\lambda)\bar{\rho}_2 \in \Phi_1 y$.

Step 2: Φ_1 maps bounded sets into bounded sets in $\mathcal{B}_h''$.

Indeed, it is enough to show that there exists a positive constant $\mathcal{K}$ such that for each $\bar{\rho} \in \Phi_1 y,\ y \in B_r = \{y \in \mathcal{B}_h'' : \|y\|_b \le r\}$, one has $\|\bar{\rho}\|_b \le \mathcal{K}$.

If $\bar{\rho} \in \Phi_1 y$, then there exists $f \in S_{F,y}$ such that for each $t \in J$, we have

$$\begin{aligned}\bar{\rho}(t) &= -T(t)p(0,\varphi) + p(t, y_t + \tilde{\varphi}_t) + \int_0^t T(t-s)f(s)ds \\ &\quad + \sum_{0<t_k<t} T(t-t_k) \times I_k(y(t_k^-) + \tilde{\varphi}(t_k^-)) \\ &\quad + \int_0^t T(t-\eta)BW^{-1}\Big\{u_1 - T(b)[\varphi(0) - p(0,\varphi] - p(b, y_b + \tilde{\varphi}_b) \\ &\quad - \int_0^b T(b-s)f(s)ds - \sum_{0<t_k<b} T(b-t_k)I_k(y(t_k^-) + \tilde{\varphi}(t_k^-))\Big\}\ (\eta)\ d\eta. \quad (3.3)\end{aligned}$$

By (H1)-(H5), (3.2) and (3,3), we have for $t \in J$

$$
\begin{aligned}
|\bar{\rho}(t)| \leq\ & M|p(0,\varphi)| + L_1 r' + L_2 + M\Omega\big[r' + L_0\Theta(r')\big]\int_0^b m(s)ds \\
& + M\sum_{k=1}^{m} d_k + bMM_1M_2 \\
& \Big[|u_1| + M[|\varphi(0) + |p(0,\varphi)|] + L_1 r' + L_2 \\
& + M\Omega\big[r' + L_0\Theta(r')\big]\int_0^b m(s)ds + M\sum_{k=1}^{m} d_k\Big] = \mathcal{K}.
\end{aligned}
$$

Thus, for each $\bar{\rho} \in \Phi_1(B_r)$, we have $\|\bar{\rho}\|_b \leq \mathcal{K}$.

Step 3: Φ_1 maps bounded sets into equicontinuous sets of $\mathcal{B}_h''$. Let $0 < \tau_1 < \tau_2 \leq b$, for each $y \in B_r = \{y \in \mathcal{B}_h'' : \|y\|_b \leq r\}$ and $\bar{\rho} \in \Phi_1 y$, there exists $f \in S_{F,y}$ satisfying (3.3).

Thus, we see that

$$
\begin{aligned}
|\bar{\rho}(\tau_2) - \bar{\rho}(\tau_1)| \leq\ & |T(\tau_2) - T(\tau_1)||p(0,\varphi)| \\
& + L(|\tau_2 - \tau_1| + \|y_{\tau_2} - y_{\tau_1}\|_{\mathcal{B}_h} + \|\varphi_{\tau_2} - \varphi_{\tau_1}\|_{\mathcal{B}_h}) \\
& + \int_0^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)||f(s)|ds + \int_{\tau_1}^{\tau_2} |T(\tau_2 - s))||f(s)|ds \\
& + \sum_{0<t_k<\tau_1} |T(\tau_2 - t_k) - T(\tau_1 - t_k)|d_k + \sum_{\tau_1<t_k<\tau_2} |T(\tau_2 - t_k)|d_k \\
& + \int_0^{\tau_1} |T(\tau_2 - \eta) - T(\tau_1 - \eta)| \times M_1M_2\Big\{|u_1| + M[\|\varphi(0)\| + |p(0,\varphi)|] + L_1 r' \\
& + L_2 + M\Omega\big[r' + L_0\Theta(r')\big]\int_0^b m(s)ds \\
& + M\sum_{k=1}^{m} d_k\Big\}\, d\eta + \int_{\tau_1}^{\tau_2} |T(\tau_2 - \eta)|M_1M_2\Big\{|u_1| + M[\|\varphi(0)\| \\
& + |p(0,\varphi)|] + L_1 r' + L_2 \\
& + M\Omega\big[r' + L_0\Theta(r')\big]\int_0^b m(s)ds + M\sum_{k=1}^{m} d_k\Big\}\, d\eta.
\end{aligned}
$$

The right hand side of above inequality is independent of $y \in B_r$ and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus the set $\{\Phi_1 y : y \in B_r\}$ is equicontinuous (Note that this proves the equicontinuity for the case where $t \neq t_k,\ k = 1, 2, \ldots, m+1$. Easily we prove the equicontinuity for the case where $t = t_i$. And also the other cases $\tau_1 < \tau_2 \leq 0$ or $\tau_1 \leq 0 \leq \tau_2 \leq b$ are very simple).

As a consequence of steps 2 and 3 together with the Arzela-Ascoli theorem we can conclude that $\Phi_1 : \mathcal{B}_h'' \to 2^{\mathcal{B}_h''}$ is a compact multivalued map, and therefore, a condensing map.

Step 4: Φ_1 has a closed graph.

Let $y_n \to y_*$, $\bar{\rho}_n \in \Phi_1 y_n$ and $\bar{\rho}_n \to \bar{\rho}_*$. We shall prove that $\bar{\rho}_* \in \Phi_1 y_*$. Indeed, $\bar{\rho}_n \in \Phi_1 y_n$ means that there exists $f_n \in S_{F,y_n}$ such that

$$\bar{\rho}_n(t) = -T(t)p(0,\varphi) + p(t, y_{n_t} + \tilde{\varphi}_t) + \int_0^t T(t-s)f_n(s)ds + \sum_{0<t_k<t} T(t-t_k)\times$$
$$I_k(y_n(t_k^-) + \tilde{\varphi}(t_k^-)) + \int_0^t T(t-s)Bx_{y_n}(s)ds, \ t \in J,$$

where

$$x_{y_n}(t) = W^{-1}\Big\{u_1 - T(b)[\varphi(0) - p(0,\varphi] - p(b, y_{n_b} + \tilde{\varphi}_b)$$
$$- \int_0^b T(b-s)f_n(s)ds - \sum_{0<t_k<b} T(b-t_k)I_k(y_n(t_k^-) + \tilde{\varphi}(t_k^-))\Big\}(t).$$

We must prove that there exists $f_* \in S_{F,y_*}$ such that

$$\bar{\rho}_*(t) = -T(t)p(0,\varphi) + p(t, y_{*_t} + \tilde{\varphi}_t) + \int_0^t T(t-s)f_*(s)ds + \sum_{0<t_k<t} T(t-t_k)\times$$
$$I_k(y_*(t_k^-) + \tilde{\varphi}(t_k^-)) + \int_0^t T(t-s)Bx_{y_*}(s)ds, \ t \in J,$$

where

$$x_{y_*}(t) = W^{-1}\Big\{u_1 - T(b)[\varphi(0) - p(0,\varphi] - p(b, y_{*_b} + \tilde{\varphi}_b)$$
$$- \int_0^b T(b-s)f_*(s)ds - \sum_{0<t_k<b} T(b-t_k)I_k(y_*(t_k^-) + \tilde{\varphi}(t_k^-))\Big\}(t).$$

Set

$$\bar{x}_y(t) = W^{-1}\Big\{u_1 - T(b)[\varphi(0) - p(0,\varphi] - p(b, y_b + \tilde{\varphi}_b)$$
$$- \sum_{0<t_k<b} T(b-t_k)I_k(y(t_k^-) + \tilde{\varphi}(t_k^-))\Big\}(t).$$

Since g, W^{-1} are continuous, then $u_{y_n}(t) \to u_{y_*}(t)$ for $t \in J$, and

$$\Big\|\Big\{\bar{\rho}_n(t) + T(t)p(0,\varphi) - p(t, y_{n_t} + \tilde{\varphi}_t) - \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-) + \tilde{\varphi}(t_k^-))$$
$$- \int_0^t T(t-s)Bx_{y_n}(s)ds\Big\} - \Big\{\bar{\rho}_*(t) + T(t)p(0,\varphi) - p(t, y_{*_t} + \tilde{\varphi}_t)$$
$$- \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-) + \tilde{\varphi}(t_k^-)) - \int_0^t T(t-s)Bx_{y_*}(s)ds\Big\}\Big\|_b$$
$$\to 0 \quad \text{as} \quad n \to \infty.$$

Consider the linear operator $\Gamma : L^1(J, X) \to C(J, X)$ defined by

$$f \to \Gamma(f)(t) = \int_0^t T(t-s)\Big[f(s) + GW^{-1}\Big(\int_0^b T(b-\tau)f(\tau)d\tau\Big)(s)\Big]ds.$$

Clearly, Γ is linear and continuous. Indeed, one has

$$\|\Gamma f\|_\infty \le (bM^2M_1M_2 + M)\|f\|_{L^1}.$$

From Lemma 3.1, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have

$$\bar{\rho}_n(t) + T(t)p(0,\varphi) - p(t, y_{n_t} + \tilde{\varphi}_t) - \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-) + \tilde{\varphi}(t_k^-))$$
$$- \int_0^t T(t-s)B\bar{x}_{y_n}(s)ds \in \Gamma(S_{F,y_n}).$$

Since $y_n \to y_*$, it follows from Lemma 3.1 that

$$\bar{\rho}_*(t) + T(t)p(0,\varphi) - p(t, y_{*_t} + \tilde{\varphi}_t) - \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-) + \tilde{\varphi}(t_k^-))$$
$$- \int_0^t T(t-s)B\bar{x}_{y_*}(s)ds$$

for some $f_* \in S_{F,y_*}$.

Hence Φ_1 is a completely continuous multivalued map, u.s.c. with convex closed values.

Now in order to apply Theorem 2.1, we introduce a parameter $\lambda > 1$ and consider the following equation:

$$\frac{d}{dt}\Big[u(t) - \frac{1}{\lambda}p(t,u_t)\Big] \in A\Big[u(t) - \frac{1}{\lambda}p(t,u_t)\Big] + \frac{1}{\lambda}F\Big(t, u_t, \int_0^t a(t,s,u_s)ds\Big) + \frac{1}{\lambda}Gx(t),$$
$$t \in J = [0,b],\ t \neq t_k, \quad k = 1,2,...,m,$$
$$\Delta u|_{t=t_k} = \frac{1}{\lambda}I_k(u(t_k^-)), \quad k = 1,2,...,m, \tag{3.4}$$
$$u_0 = \varphi \in \mathcal{B}_h.$$

Thus, by Definition 3.1, the mild solution of (3.4) can be written as

$$
\begin{aligned}
u(t) = {} & T(t)[\varphi(0) - \frac{1}{\lambda} p(0,\varphi)] + \frac{1}{\lambda} p(t,u_t) \\
& + \frac{1}{\lambda}\int_0^t T(t-s) f(s) ds + \frac{1}{\lambda} \sum_{0<t_k<t} T(t-t_k) I_k(u(t_k^-)) \\
& + \frac{1}{\lambda}\int_0^t T(t-\eta) G W^{-1}\Big\{ u_1 - T(b)[\varphi(0) - p(0,\varphi] - p(b,u_b) \\
& - \int_0^b T(b-s) f(s) ds \\
& - \sum_{0<t_k<b} T(b-t_k) I_k(u(t_k^-)) \Big\}(\eta) d\eta, \quad t \in J, \\
& \varphi(t), \quad t \in (-\infty, 0].
\end{aligned} \tag{3.5}
$$

where

$$
f \in S_{F,u} = \left\{ f \in L^1(J,X) : f(t) \in F\Big(t, u_t, \int_0^t a(t,s,u_s) ds\Big), \text{ for a.e. } t \in J \right\}.
$$

■

Lemma 3.7. If hypotheses (H1)-(H6) are satisfied, let $u(t)$ be a mild solution of system (3.4), then there exists a priori bound $\mathbb{K} > 0$ such that $\|u_t\|_{\mathcal{B}_h} \leq \mathbb{K},\ t \in J$, where $\mathbb{K}$ depends only on b and on the functions $m(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$.

Proof. From equation (3.5), we obtain

$$
\begin{aligned}
|u(t)| \leq {} & M[|\varphi(0)| + L_1\|\varphi\|_{\mathcal{B}_h} + L_2] + L_1\|u_t\|_{\mathcal{B}_h} + L_2 \\
& + M\int_0^t \Omega\big[\|u_s\|_{\mathcal{B}_h} + L_0\Theta(\|u_\tau\|_{\mathcal{B}_h})\big] m(s) ds \\
& + M\sum_{k=1}^m d_k + MbM_1M_2\Big[|u_1| + M[|\varphi(0)| + L_1\|\varphi\|_{\mathcal{B}_h} + L_2] \\
& + L_1\|u_b\|_{\mathcal{B}_h} + L_2 \\
& + M\int_0^b \Omega\big[\|u_s\|_{\mathcal{B}_h} + L_0\Theta(\|u_\tau\|_{\mathcal{B}_h})\big] m(s) ds + M\sum_{k=1}^m d_k\Big] \\
& \leq M[|\varphi(0)| + L_1\|\varphi\|_{\mathcal{B}_h} + L_2] + L_1\|u_t\|_{\mathcal{B}_h} + L_2
\end{aligned}
$$

$$\begin{aligned}
&+ M\int_0^t \Omega\big[\|u_s\|_{\mathcal{B}_h} + L_0\Theta(\|u_\tau\|_{\mathcal{B}_h})\big]m(s)ds \\
&+ M\sum_{k=1}^{m} d_k + MbN \\
\le\ & K_1 + L_1\|u_t\|_{\mathcal{B}_h} + M\int_0^t \Omega\big[\|u_s\|_{\mathcal{B}_h} + L_0\Theta(\|u_\tau\|_{\mathcal{B}_h})\big]m(s)ds, \quad t\in J.
\end{aligned}$$

Thus from this proof and Lemma 3.2, we get

$$\begin{aligned}
\|u_t\|_{\mathcal{B}_h} &\le \ell\sup\{|u(s)| : 0\le s\le t\} + \|\varphi\|_{\mathcal{B}_h} \le \|\varphi\|_{\mathcal{B}_h} + \ell K_1 + \ell L_1 \sup_{0\le s\le t}\|u_s\|_{\mathcal{B}_h} \\
&+ \ell M\int_0^t m(s)\Omega\Big(\|u_s\|_{\mathcal{B}_h} + \int_0^s \gamma p(\tau)\Theta(\|u_\tau\|_{\mathcal{B}_h})d\tau\Big)ds,\ t\in J.
\end{aligned}$$

Let $\mu(t) = \sup\{\|u_s\|_{\mathcal{B}_h} : 0\le s\le t\}$, then the function $\mu(t)$ is nondecreasing in J, and we have

$$\mu(t) \le h_1 + h_2\int_0^t m(s)\Omega\Big(\mu(s) + \int_0^s \gamma p(\tau)\Theta(\mu(\tau))d\tau\Big)ds,\ t\in J.$$

Denoting by the right hand side of the above inequality as $v(t)$, we see that $v(0) = h_1,\ \ \mu(t)\le v(t),\ t\in J$ and

$$v'(t) = h_2\Omega\Big(\mu(t) + \int_0^t \gamma p(s)\Theta(\mu(s))ds\Big).$$

Since Ω is nondecreasing

$$v'(t) \le h_2\Omega\Big(v(t) + \int_0^t \gamma p(s)\Theta(v(s))ds\Big),\ t\in J.$$

Let

$$w(t) = v(t) + \int_0^t \gamma p(s)\Theta(v(s))ds.$$

Then $w(0) = v(0)\quad and\quad v(t)\le w(t)$.

$$\begin{aligned}
w'(t) &= v'(t) + \gamma p(t)\Theta(v(t)) \le h_2\Omega(w(t)) + \gamma p(t)\Theta(v(t)) \\
&\le \widetilde{m}(t)\big[\Omega(w(t)) + \Theta(w(t))\big].
\end{aligned}$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega(s)+\Theta(s)} \le \int_0^b \widetilde{m}(s)ds < \int_{h_1}^{\infty}\frac{ds}{\Omega(s)+\Theta(s)}, \quad t\in J.$$

This inequality implies that $w(t) < \infty$. Hence there is a constant $\mathbb{K}$ such that $w(t) \leq \mathbb{K}$, $t \in J$. Thus, we have $\|u_t\|_{\mathcal{B}_h} \leq \mu(t) \leq v(t) \leq w(t) \leq \mathbb{K}$, $t \in J$, where $\mathbb{K}$ depends only on b and on the functions $m(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$. ■

Theorem 3.8. Assume that the hypotheses (H1)-(H6) hold. Then system (1.1)-(1.3) is controllable on J.

Proof. Let $\Lambda = \{y \in \mathcal{B}_h'' : \lambda y \in \Phi_1 y \quad \text{for some } \lambda \in (0,1)\}$. Then for any $y \in \Lambda$, we have

$$
\begin{aligned}
u(t) = & -T(t)\frac{1}{\lambda}p(0,\varphi) + \frac{1}{\lambda}p(t, y_t + \tilde{\varphi}_t) + \frac{1}{\lambda}\int_0^t T(t-s)f(s)ds \\
& + \frac{1}{\lambda}\sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-) + \tilde{\varphi}(t_k^-)) \\
& + \frac{1}{\lambda}\int_0^t T(t-\eta)GW^{-1}\Big\{u_1 - T(b)[\varphi(0) - p(0,\varphi] \\
& - p(b, y_b + \tilde{\varphi}_b) - \int_0^b T(b-s)f(s)ds \\
& - \sum_{0<t_k<b} T(b-t_k)I_k(y(t_k^-) + \tilde{\varphi}(t_k^-))\Big\}(\eta)d\eta, \ t \in J,
\end{aligned}
$$

which implies the function $u = y + \tilde{\varphi}$ is a mild solution of above system (3.4), for which we have proved in Lemma 3.4 as $\|u_t\|_{\mathcal{B}_h} \leq \mathbb{K}$, $t \in J$, and hence from Lemma 3.2 and [20] we have

$$\|y\|_b \leq l^{-1}\mathbb{K} + M|\varphi(0)|$$

which implies that the set Λ is bounded on J.

Hence it follows from Lemma 3.3 and Theorem 2.1 that the operator Φ_1 has a fixed point $y_* \in \mathcal{B}_h''$. Let $u(t) = y_*(t) + \tilde{\varphi}(t)$, $t \in (-\infty, b]$. Then u is a fixed point of the operator Φ which is a mild solution of the problem (1.1)-(1.4). ■

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