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A Fixed Point of A Self Mapping In Hilbert Space

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Abstract

In this paper, It is aimed to examine the existence and uniqueness of fixed point of a self mapping on a closed subset of Hilbert space which satisfies inequality involving rational term. In addition to this,the fruitful result of Dass and Gupta in Hilbert space is obtained by assigning the vanishing values to the constants in the result.

Keywords: Cauchy sequence, Closed subset, Completeness, Hilbert space, Uniqueness.

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Introduction

First the existence and unique fixed point was introduced by the mathematician Banach in 1922, which was acclaimed as Banach contraction principle.It is known fact that it plays an important role in the development of various results connected with Fixed point Theory and Approximation Theory. The Banach's fixed point theorem or the contraction principle concerns certain mappings of a complete metric space into itself. It lays down conditions; sufficient for the existence and uniqueness of a fixed point. Besides, this famous classical theorem gives an iteration process through which we can obtain better approximation to the fixed point; Banach's fixed point theorem has rendered a key role in solving systems of linear algebraic equations involving iteration process. Iteration procedures are used in nearly every branch of applied mathematics, and convergence proof and error estimates are very often obtained by the application of Banach's fixed point theorem. After that several mathematicians contributed to the growth of this area of knowledge as has been extensively reported in the treatises of Nadler [4], Sehgal [6], Kannan [2], Reich [5] Zamfirescu [10], Wong [9], Smart [8], Koparde and Waghmode [3], S.P. Singh, E. Rakotc and many other authors.

The result which is found here is the refinement and sharpens some of the generalizations of Singh. Th. Manihar [7], Smart [8] and Dass and Gupta [1] results of Banach's celebrated theorems. The theorem follows with the statement

Theorem: *Let X be a closed subset of a Hilbert space and* $T : X \to X$ *be a self mapping satisfying the following condition*

$$\|Tx - Ty\| \le a_1 \frac{\|y - Ty\| \,[1 + \|x - Tx\|]}{1 + \|x - y\|} + a_2[\|x - Tx\| + \|y - Ty\|]$$

$$+ a_3[\|x - Ty\| + \|y - Tx\|] + a_4\|x - y\|$$

for all $x, y \in X$ *and* $x \ne y$ *, where* a_1, a_2, a_3, a_4 *are non-negative reals with* $0 \le a_1 + 2(a_2 + a_3) + a_4 < 1$. *Then* T *has a unique fixed point in* X *.*

Proof : For any arbitrary point $x_0 \in X$, define a sequence $\{x_n\}$ by

$x_{n+1} = Tx_n$, for $n = 0,1,2,3,........$

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in X . For that, it is observed the following

$$\|x_{n+1} - x_n\| = \|Tx_n - Tx_{n-1}\|$$

Then by using the given condition, we have

$$\|x_{n+1} - x_n\| \le a_1 \frac{\|x_{n-1} - Tx_{n-1}\| \,[1 + \|x_n - Tx_n\|]}{1 + \|x_n - x_{n-1}\|} + a_2[\|x_n - Tx_n\| + \|x_{n-1} - Tx_{n-1}\|]$$

$$+ a_3[\|x_n - Tx_{n-1}\| + \|x_{n-1} - Tx_n\|] + a_4\|x_n - x_{n-1}\|$$

which implies that

$$(1 - a_2 - a_3)\|x_{n+1} - x_n\| + (1 - a_1 - a_2 - a_3)\|x_{n+1} - x_n\| \; \|x_n - x_{n-1}\|$$

$$\le \{(a_1 + a_2 + a_3 + a_4) + (a_2 + a_3 + a_4) \; \|x_n - x_{n-1}\|\} \; \|x_n - x_{n-1}\|$$

and so $\|x_{n+1} - x_n\| \le p(n)\|x_n - x_{n-1}\|$

where $p(n) = \dfrac{(a_1 + a_2 + a_3 + a_4) + (a_2 + a_3 + a_4)\|x_n - x_{n-1}\|}{(1 - a_2 - a_3) + (1 - a_1 - a_2 - a_3)\|x_n - x_{n-1}\|}$, for $n = 1,2,3,........$

Clearly $p(n) < 1$ as $0 \le a_1 + 2(a_2 + a_3) + a_4 < 1$. Proceeding in this way, we can get some $S < 1$ such that $\|x_{n+1} - x_n\| \le S^n \|x_1 - x_0\|$

On letting $n \to \infty$, we get $\|x_{n+1} - x_n\| \to 0$. It follows that the sequence $\{x_n\}$ is a Cauchy sequence in X. Using the completeness of X there is some $\mu \in X$ such that $x_n \to \mu$ as $n \to \infty$.

Also, $\{x_{n+1}\} = \{Tx_n\}$ is a subsequence of $\{x_n\}$ and hence it has the same limit. Since T is continuous, we have

$$T(\mu) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = \mu$$

Thus T has a fixed point μ in X. To show that μ is a unique fixed point of T, let us suppose that $v(\mu \neq v)$ is also a fixed point of T; i.e. $Tv = v$. Then

$$\|\mu - v\| \neq 0$$

Now, we have the following inequality; namely;

$$\|\mu - v\| = \|T\mu - Tv\|$$

$$\leq a_1 \frac{\|v - Tv\| \, [1 + \|\mu - T\mu\|]}{1 + \|\mu - v\|} + a_2 [\|\mu - T\mu\| + \|v - Tv\|]$$

$$+ a_3 [\|\mu - Tv\| + \|v - T\mu\|] + a_4 \|\mu - v\|$$

which implies that $\|\mu - v\| \leq (2a_3 + a_4)\|\mu - v\|$

This results in a contradiction for $2a_3 + a_4 < 1$. Therefore $\mu = v$. Thus, μ is a unique fixed point of T in X.

NOTE : On putting $a_2 = a_3 = 0$, the above result reduces to the Dass and Gupta's result in a Hilbert space.

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