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Sufficient Conditions for Oscillation of First Order Neutral Delay Difference Equations with Variable Coefficients

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Abstract

In this paper, we obtain some new sufficient conditions for oscillation of all solutions of the first order linear neutral delay difference equations are established. Our new results improve many well known results in the literature. Some examples are considered to illustrate our results.
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Introduction

During the last several years many research papers on the oscillatory behavior of solutions of neutral delay difference equations have appeared in the literature, as these equations occur as mathematical models of some real world problems, see [4,8].

In this paper, we consider the linear first-order neutral delay difference equation of the form

$$\Delta\left[\, r(n)\left(x(n) + p(n)x(n-\tau)\right)\right] + q(n)x(n-\sigma) = 0, \qquad n \geq n_0, \tag{1}$$

where Δ is the forward difference operator given by $\Delta x(n) = x(n+1) - x(n)$, $\{p(n)\}$ is a sequence of real numbers, $\{r(n)\}$, $\{q(n)\}$ are sequences of positive real numbers and τ, σ are positive integers.

Let us choose a positive integer $n^* > \max\{\tau, \sigma\}$. By a solution of (1) on $N(n_0) = \{n_0, n_0 + 1, \dots\}$, we mean a nontrivial real sequence $\{x(n)\}$ which is defined on $n \geq n_0 - n^*$ and which satisfies (1) for $n \in N(n_0)$. A solution $\{x(n)\}$ of (1) on $N(n_0)$ is said to be oscillatory if for every positive integer $N_0 > n_0$ there exists $n \geq N_0$ such that $x(n)\, x(n+1) \leq 0$, otherwise $\{x(n)\}$ is said to be nonoscillatory.

Recently some investigations have appeared which are concerned with the oscillation as well as the nonoscillation behavior of (1). For the oscillation of (1) when $r(n) \equiv 1$ and p(n) and q(n) are constants, we refer the readers to the articles by D. A. Georgiu et al. [5] and Györi and Ladas [6]. For the oscillation of (1) when r(n) = 1 and p(n) is equal to a constant, we refer the readers to the paper by B. S. Lalli [7] and the references cited there in. For further oscillation results on the oscillatory behavior of solutions of (1), we refer the readers to the monographs by R. P. Agarwal [1, 2] as well as the papers of Ying Gao and Guang Zhang [11], M. P. Chen et al. [3], X. H. Tang et al. [10] and Ö. Öcalan and O. Duman [9] and the references cited there in. The purpose of this work is to find some sufficient conditions for oscillations of all solutions of the first order neutral delay difference equation (1).

Remark 1.1.

I. *When we write a functional inequality we assume that it holds for all sufficiently large n.*
II. *Without loss of generality, we will deal only with the eventually positive solutions of (1).*

In the proof of our main results, we need the following lemmas. The Lemmas 1.4, 1.5 and 1.6 are the discrete analogues of the Lemmas 1.5.1, 1.5.2, and 1.5.3 respectively in [6].

Lemma 1.2. [6] Assume that k is a positive integer with k > 1. Let {h(n)} be a sequence of nonnegative real numbers and suppose that

$$\liminf_{n\to\infty} \sum_{s=n-k}^{n-1} h(s) > \left(\frac{k}{k+1}\right)^{k+1} \tag{2}$$

Then

(i) *the delay difference inequality*

$$\Delta x(n) + h(n)x(n-k) \leq 0, \quad n \geq n_0, \tag{3}$$

has no eventually positive solution.

(ii) *the delay difference inequality*

$$\Delta x(n) + h(n)x(n-k) \geq 0, \quad n \geq n_0, \tag{4}$$

has no eventually negative solution.

Lemma 1.3. [6] *Assume that k is a positive integer with k > 1. Let {h(n)} be a sequence of nonnegative real numbers and suppose that*

$$\liminf_{n\to\infty} \sum_{s=n+1}^{n+k-1} h(s) > \left(\frac{k-1}{k}\right)^{k}. \tag{5}$$

Then

(i) the advanced difference inequality

$$\Delta x(n) - h(n)x(n+k) \le 0, \quad n \ge n_0, \tag{6}$$

has no eventually negative solution.

(ii) the advanced difference inequality

$$\Delta x(n) - h(n)x(n+k) \ge 0, \quad n \ge n_0, \tag{7}$$

has no eventually positive solution.

Lemma 1.4. *Let {f(n)} and {g(n)} be sequence of real numbers such that*

$$f(n) = g(n) + \mu g(n-c); \quad n \ge n_0 + \max\{0,c\},$$

where $\mu \in R$, $\mu \ne 1$ and c is a positive integer. Assume that $\lim_{n\to\infty} f(n) = l \in R$ exist and $\lim\ \inf_{n\to\infty} g(n) = a \in R$. Then $l = (1+p)a$.

Lemma 1.5. *Let {f(n)}, {g(n)} and {λ(n)} be sequences of real numbers and c is a positive integer such that*

$$f(n) = g(n) + \lambda(n)g(n-c); \quad \text{for} \quad n \ge n_0 + \max\{0,c\},$$

Assume that $-1 \le \lambda(n) \le 0$. Suppose that $g(n) > 0$ for $n \ge n_0$, $\lim\ \inf_{n\to\infty} g(n) = 0$ and that $\lim_{n\to\infty} f(n) = L \in R$ exist. Then $L = 0$.

Lemma 1.6. *Let $0 \le \lambda < 1$, c be a positive integer and $n_0 \in N$ and {x(n)} be a sequence of positive real numbers and assume that for every $\varepsilon > 0$ there exists a $n_\epsilon \ge n_0$ such that*

$$x(n) \le (\lambda + \varepsilon)\, x(n-c) + \varepsilon \quad \text{for} \quad n \ge n_\epsilon.$$

Then

$$\lim_{n\to\infty} x(n) = 0.$$

Lemma 1.7. *Assume that*

$$\sum_{n=n_0}^{\infty} q(n) = \infty. \tag{8}$$

Let $\{x(n)\}$ be an eventually positive solution of equation (1). Set

$$z(n) = x(n) + p(n)x(n-\tau). \tag{9}$$

Then the following statements are true.

(i) if $p(n) \le -1$ then $z(n) < 0$;

(ii) if $-1 \le p(n) \le 0$ and {r(n)} is a decreasing sequence of positive real numbers, then $z(n) > 0$ and $\lim_{n\to\infty} z(n) = 0$.

Proof. (i) Otherwise, $z(n) > 0$ and so

$$x(n) \ge -p(n)x(n-\tau) \ge x(n-\tau)$$

which implies that $\{x(n)\}$ is bounded from below by a positive constant, say m. From (1) and (9), we have

$$\Delta(r(n)z(n)) = -q(n)x(n-\sigma) \le -mq(n) \tag{10}$$

which in view of (8) implies that

$$\lim_{n\to\infty}\big(r(n)z(n)\big) = -\infty.$$

This is a contradiction and so the proof of (i) is complete.

(i) First we claim that z(n) > 0. Otherwise, z(n) < 0, which implies that

$$x(n) < -p(n)x(n-\tau) \leq x(n-\tau)$$

and so $\{x(n)\}$ is bounded sequence. Hence $\{z(n)\}$ is also bounded. Since $\{r(n)\}$ is decreasing sequence, $\{z(n)\}$ is also decreasing sequence. Then both the limits $\lim_{n\to\infty}(r(n)z(n))$ and $\lim_{n\to\infty} z(n)$ exists.

Let

$$\lim_{n\to\infty}\big(r(n)z(n)\big) = l_1 \in R, \tag{11}$$

and

$$\lim_{n\to\infty} z(n) = l_2 \in R. \tag{12}$$

But then because of (8),

$$\lim\ \inf_{n\to\infty} x(n) = 0. \tag{13}$$

Otherwise, by summing (10) from n_1 to ∞, with n_1 sufficiently large, we are lead to the contradiction that

$$\lim_{n\to\infty}\big(r(n)z(n)\big) = -\infty.$$

In view of (12) and (13), by applying Lemma 1.5, we find that l_2=0. This implies that z(n) >0, which contradicts our hypothesis that z(n)<0. Therefore we have established that z(n) >0. But then (12) and (13) are also true and by Lemma 1.4, $l_2 = 0$. The proof of (ii) is complete.

Lemma 1.8. *Assume that (8) holds and let { x(n)} be an eventually positive solution of the neutral difference equation*

$$\Delta[x(n) + px(n-\tau)] + q(n)x(n-\sigma) = 0, \quad n \leq n_0, \tag{14}$$

where p is a real numbers with p ≠ 1, { q (n)} is a τ – periodic sequence of positive real numbers and τ and σ are positive integers. Set

$$z(n) = x(n) + p\,x(n-\tau) \tag{15}$$

and

$$w(n) = z(n) + pz(n-\tau). \tag{16}$$

Then

{z(n)} is a decreasing sequence and either

$$\lim_{n\to\infty} z(n) = -\infty, \tag{17}$$

or

$$\lim_{n\to\infty} z(n) = 0. \tag{18}$$

(a) The following statements are equivalent:

(i) *(17) holds;*
(ii) *$P < -1$;*
(iii) *$lim_{n\to\infty} x(n) = \infty$;*
(iv) *$w(n) > 0$, $\Delta w(n) > 0$.*

(b) The following statements are equivalent :
(i) *(18) holds;*
(ii) *$P > -1$;*
(iii) *$lim_{n\to\infty} x(n) = 0$;*
(iv) *$w(n) > 0$, $\Delta w(n) > 0$.*

Proof. (a) We have

$$\Delta z(n) = -q(n)x(n-\sigma) < 0 \tag{19}$$

and so $\{z(n)\}$ is strictly decreasing sequence. If (17) is not true, then there exits $l \in$ R such that $\lim_{n\to\infty} z(n) = l$. By summing (19) from n_1 to ∞, with n_1 sufficiently large, we find

$$l - z(n_1) = -\sum_{s=n_1}^{\infty} q(s)x(s-\sigma). \tag{20}$$

In view of (8) this implies that $\lim\inf_{n\to\infty} x(n) = 0$ and so by Lemma 1.4, l = (1+p) 0 = 0. The proof of (a) is complete.

Now we turn to the proofs of (b) and (c). First assume that (17) holds. Then p must be negative and $\{x(n)\}$ is unbounded. Therefore there exists an integer $n^* \geq n_0$ such that $z(n^*) < 0$ and

$$x(n^*) \geq \max_{s\leq n^*} x(s) > 0.$$

Then

$$\begin{aligned} 0 > z(n^*) &= x(n^*) + px(n^*-\tau) \\ &\geq x(n^*)(1+p) \end{aligned}$$

which implies that $p < -1$. Also $z(n) = x(n) + px(n-\tau) > px(n-\tau)$ and (17) implies that $\lim_{n\to\infty} x(n) = \infty$. Now assume that (18) holds. If $p \geq 0$, then from (15), it follows that $\lim_{n\to\infty} x(n) = 0$. Next assume that $p \in (-1,0)$. Then by Lemma 1.6, $\lim_{n\to\infty} x(n) = 0$. Finally if $p \leq -1$, then $x(n) > -px(n-\tau) \geq x(n-\tau)$ which shows that $\{x(n)\}$ is bounded from below by a positive constant, say m. Then (20) yields.

$$l - z(n_1) + m\sum_{s=n_1}^{\infty} q(s) \leq 0,$$

which is a contradiction. Therefore, if (18) holds, $p > -1$. Observe that under the hypothesis (17), we have

$$\Delta w(n) = -q(n)z(n-\sigma) > 0. \tag{21}$$

If (17) holds, then

$$\lim_{n\to\infty} w(n) = \infty. \tag{22}$$

From (21) and (22), we have w(n) > 0 eventually. By a similar proof, under the hypothesis (18), we have Δw(n) <0 and w(n) > 0. On the basis of the above discussions, the proofs of (b) and (c) are now obvious.

Main Results

In this section we give some new sufficient conditions for all solutions of the neutral delay difference equation (1) to be oscillatory.

Theorem 2.1. *Assume that (8) holds, p(n) ≤ -1, τ- σ > 1, and*

$$\liminf_{n\to\infty} \sum_{s=n+1}^{n+\tau-\sigma-1} \left\{\frac{q(s)}{-r(s+\tau-\sigma)p(s+\tau-\sigma)}\right\} > \left(\frac{\tau-\sigma-1}{\tau-\sigma}\right)^{\tau-\sigma}.$$

Then every solution of (1) is oscillatory.

Proof. Assume, for the sake of a contradiction, that (1) has an eventually positive solution $\{x(n)\}$. Set

$$z(n) = x(n) + p(n)x(n-\tau). \tag{23}$$

Then by Lemma 1.7 (i),

$$z(n) < 0. \tag{24}$$

Observe that

$$z(n) > p(n)x(n-\tau). \tag{25}$$

From which we find eventually

$$\Delta\big(r(n)z(n)\big) + \frac{q(n)}{p(n+\tau-\sigma)} z(n+\tau-\sigma) < 0. \tag{26}$$

Set $y(n) = r(n)z(n)$. This implies that $y(n) < 0$. Substituting in (26) yields

$$\Delta y(n) + \frac{q(n)}{r(n+\tau-\sigma)p(n+\tau-\sigma)} y(n+\tau-\sigma) < 0, \tag{27}$$

Or

$$\Delta y(n) - \left(\frac{q(n)}{-r(n+\tau-\sigma)p(n+\tau-\sigma)}\right) y(n+\tau-\sigma) < 0. \tag{28}$$

In view of (22) and Lemma 1.3 (i), it is impossible for (28) to have an eventually negative solution. This contradicts the fact that y(n) < 0 and the proof is complete.

Example 2.2. *Consider the first order neutral delay difference equation*

$$\Delta\left[\frac{e^n}{n+1}\left(x(n) - \left(\frac{n+1}{n}\right)x(n-3)\right)\right] + e^{n+2}x(n-1) = 0, \quad n = 1,2,3,\ldots. \tag{E_1}$$

Here we have

$$p(n) = -\frac{n+1}{n} \le -1, \quad q(n) = e^{n+2},$$

$$r(n) = \frac{e^n}{n+1}, \qquad \tau = 3 \, and \, \sigma = 1.$$

Then all the hypotheses of Theorem 2.1 are satisfies where

$$\liminf_{n\to\infty} \sum_{s=n+1}^{n+\tau-\sigma-1} \left\{ \frac{q(s)}{-r(s+\tau-\sigma)p(s+\tau-\sigma)} \right\} = \liminf_{n\to\infty} (n+2) = \infty.$$

Hence every solution of (E_1) is oscillatory

Remark 2.3. *Theorem 2.1 is an extent of [6, Theorem 7.5.1].*

Theorem 2.4. *Assume that (8) hold, $-1 \le p(n) \le 0$, $\{r(n)\}$ is a decreasing sequence of positive real numbers and*

$$\liminf_{n\to\infty} \sum_{s=n-\sigma}^{n-1} \frac{q(s)}{r(s-\sigma)} > \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1}. \tag{29}$$

Then every solution of equation (1) oscillates.

Proof. Assume, for the sake of contradiction, that (1) has an eventually positive solution $\{x(n)\}$. Set

$$z(n) = x(n) + p(n)x(n). \tag{30}$$

Then by Lemma 1.7 (ii), it follows that

$$z(n) > 0. \tag{31}$$

As $x(n) > z(n)$, it follows from (1), that

$$\Delta\big(r(n)z(n)\big) + q(n)z(n-\sigma) \le 0. \tag{32}$$

Set

$$y(n) = r(n)z(n). \tag{33}$$

This implies that y(n) > 0. From (32) and (33), we have

$$\Delta y(n) + \frac{q(n)}{r(n-\sigma)} y(n-\sigma) \le 0. \tag{34}$$

In view of Lemma 1.2 (i), it is impossible for (34) to have an eventually positive solution. This contradicts the fact that y(n) > 0 and the proof is complete.

Example 2.5. *Consider the first order neutral delay difference equation*

$$\Delta\left[\frac{1}{n+1}\left(x(n) - \frac{n}{n+1}x(n-2)\right)\right] + \frac{1}{n}\, x(n-1) = 0, \qquad n = 1,2,3,\dots \qquad (E_2)$$

Note that all the hypotheses of Theorem 2.4 are satisfied:

$$\liminf_{n\to\infty} \sum_{s=n-\sigma}^{n-1} \frac{q(s)}{r(s-\sigma)} = 1 > \frac{1}{4}.$$

Therefore every solution of (E_2) is oscillatory.

Remark 2.6. *Theorem 2.4 is an extent of [6, Theorem 7.5.1]*

Theorem 2.7. *Assume that $p(n) \equiv p > -1$, $r(n) \equiv r > 0$, $q(n)$ is a τ periodic , $\sigma > \tau$ and*

$$\frac{1}{r(1+p)} \liminf_{n\to\infty} \sum_{s=n-\sigma+\tau}^{n-1} q(s) > \left(\frac{\sigma-\tau}{\sigma-\tau+1}\right)^{\sigma-\tau+1}. \quad (35)$$

Then every solution of equation (1) is oscillatory.

Proof. Assume, for the sake of contradiction, that (1) has an eventually positive solution $\{x(n)\}$. Set

$$z(n) = x(n) + px(n-\tau)$$

And

$$w(n) = z(n) + pz(n-\tau).$$

It is easily seen, by direct substituting, that {z(n)} and { w(n)} are also solutions of (35).
That is,

$$r\Delta z(n) + pr\Delta z(n-\tau) + q(n)z(n-\sigma) = 0, \quad (36)$$

And

$$r\Delta w(n) + pr\Delta w(n-\tau) + q(n)w(n-\sigma) = 0. \quad (37)$$

By Lemma 1.8, $\{z(n)\}$ is decreasing $w(n) > 0$ and $\Delta w(n) > 0$. We claim that

$$\Delta w(n-\tau) \le \Delta w(n). \quad (38)$$

Indeed,

$$\Delta w(n) = \frac{-1}{r} q(n)z(n-\sigma) \ge \frac{-1}{r} q(n)z(n-\sigma-\tau)$$
$$\frac{-1}{r} q(n-\tau)z(n-\sigma-\tau)$$
$$= \Delta w(n-\tau).$$

Using (38) in (37) implies

$$r(1+p)\Delta w(n-\tau) + q(n)w(n-\sigma) \le 0, \quad (39)$$

Or

$$\Delta w(n-\tau) + \frac{q(n)}{r(1+p)} w(n-\sigma) \le 0. \quad (40)$$

Since q(n) is a periodic of period τ , we find

$$\Delta w(n) + \frac{q(n)}{r(1+p)} w(n-(\sigma-\tau)) \leq 0. \quad (41)$$

In view of Lemma 1.2 (i) and (35), it is impossible for (41) to have an eventually positive solution. This contradicts the fact that $w(n) > 0$ and the proof is complete.

Theorem 2.8. Assume that $p(n) \equiv p < -1$, $r(n) \equiv r > 0$, $q(n)$ is a τ periodic, $\tau - \sigma > 1$ and

$$\frac{1}{-r(1+p)} \liminf_{n\to\infty} \sum_{s=n+1}^{n+\tau-\sigma-1} q(s) > \left(\frac{\tau-\sigma-1}{\tau-\sigma}\right)^{\tau-\sigma} .$$

Then every solution of (1) is oscillatory.

Proof of the theorem is similar to that of the Theorem 2.7.

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