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Contractive modulus and common fixed point for two pairs of weakly compatible self-maps

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Abstract

Let S, T and A be self-maps on a metric space (X, d) satisfying the inclusions $S(X) \subset A(X)$ and $T(X) \subset A(X)$ and the inequality

$$d(Sx, Ty) \leq \phi\Big(\max\Big\{d(Ax, Ay), d(Sx, Ax), d(Ty, Ay), \frac{d(Ty, Ax)+d(Sx, Ay)}{2}\Big\}\Big) \quad (1)$$

for all $x, y \in X$, where ϕ is an upper semicontinuous contractive modulus with $\phi(0) = 0$ and $\phi(t) < t$ whenever $t > 0$. Singh and Mishra (1997) proved that if any one of the subspaces $S(X)$, $T(X)$ and $A(X)$ of X is complete, then S, T and A will have a common coincidence point. Further if the pairs (A, S) and (A, T) commute at their coincidence points, that is (A, S) and (A, T) are weakly compatible pairs, then S, T and A will have a unique common fixed point.

The present paper extends the above result to four self-maps under weaker form of the inequality (1). It can also be shown that the weak compatibility of either of the pairs (A, S) and (A, T) is sufficient to obtain a common fixed point for the three maps.

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1. Introduction

In this paper, (X, d) denotes a metric space, Sx the image of $x \in X$ under a self-map S on X and SA, the composition of self-maps S and A on X.

Definition 1.1. Self-maps S and A on X are compatible [1] if

$$\lim_{n\to\infty} d(SAx_n, ASx_n) = 0 \tag{1.1}$$

whenever $\langle x_n\rangle_{n=1}^{\infty} \subset X$ is such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = p \quad \text{for some} \quad p \in X. \tag{1.2}$$

If $x_n = x$ for all n, compatibility of (S, A) implies that $SAx = ASx$ whenever $Ax = Sx$. Self-maps which commute at their coincidence points are weakly compatible [2].

Definition 1.2. Let $\phi \equiv \phi : [0, \infty) \to [0, \infty)$ be a contractive moduluswith the choice $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$. A contractive modulus ϕ is upper semicontinuous (abbriviated as usc) if and only if

$$\limsup_{n\to\infty} \phi(t_n) \le \phi(t_0)$$

for all $t = t_0$ and all

$$\langle t_n\rangle_{n=1}^{\infty} \subset [0, \infty)$$

with

$$\lim_{n\to\infty} t_n = t_0.$$

Using these ideas, Singh and Mishra [3] proved the following result:

Theorem 1.3. Let S, T and A be self-maps on X satisfying the inclusions

$$S(X) \subset A(X) \quad \textit{and} \quad T(X) \subset A(X) \tag{1.3}$$

and the contractive-type condition

$$d(Sx, Ty) \le \phi\left(\max\left\{d(Ax, Ay), d(Sx, Ax), d(Ty, Ay), \frac{d(Ty, Ax) + d(Sx, Ay)}{2}\right\}\right) \quad \text{for all} \quad x, y \in X, \tag{1.4}$$

where ψ is an usc contractive modulus. Suppose that

(a) one of $S(X)$, $T(X)$ and $A(X)$ is a complete subspace of X,

(b) (A, S) and (A, T) are weakly compatible.

Then the three maps S, T and A will have a unique common fixed point.

The present paper extends Theorem 1.3 to two pairs of weakly compatible self-maps under weaker form of the inequality (1.4). As a particular case for three self-maps A, S and T, we obtain its corollary which reveals that the weak compatibility of either of the pairs (A, S) and (A, T) is sufficient to obtain a common fixed point for the three maps.

2. Main Result

We prove the following result

Theorem 2.1. Let A,B, S and T be self-maps on X satisfying the inclusions

$$S(X) \subset A(X) \quad and \quad T(X) \subset B(X) \tag{2.1}$$

and the inequality

$$\begin{aligned} d(Sx, Ty) \leq \phi\Bigg(\max \Bigg\{ & d(Ax, By), d(Ax, Sx), d(By, Ty), \frac{d(Ty, Ax) + d(Sx, By)}{2}, \\ & \frac{1}{4} \cdot \frac{d(Ax, Ty) + d(Ax, Sx) + d(Sx, By)}{1 + d(By, Ty)d(Sx, By)d(Ax, Ty)}, \\ & \frac{1}{4} \cdot \frac{d(Ax, Ty) + d(Sx, By) + d(By, Ty)}{1 + d(Ax, Ty)d(Ax, Sx)d(Sx, By)} \Bigg\} \Bigg) \\ & for\ all\ x, y \in X, \end{aligned} \tag{2.2}$$

where ϕ is usc contractive modulus. Suppose that any one of the subspaces $S(X)$, $T(X)$ and $A(X) \bigcup B(X)$ of X is complete. Then S, T, A and B will have a common coincidence point. Further if (A, S) and (B, T) are weakly compatible, then S, T, A and B will have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Using the inclusions (2.1), we can inductively choose points $x_1, x_2, \ldots, x_n, \ldots$ in X such that

$$y_{2n-1} = Sx_{2n-2} = Bx_{2n-1}, \, y_{2n} = Tx_{2n-1} = Ax_{2n}, \text{ for } n = 1, 2, 3, ... \tag{2.3}$$

We first show that

$$\lim_{n \to \infty} d(y_n, y_{n+1}) = 0. \tag{2.4}$$

Now wirting $x = x_{2n}$ and $y = x_{2n-1}$ in (2.2), we see that

$$\begin{aligned} d(y_{2n+1}, y_{2n}) &= d(Sx_{2n}, Tx_{2n-1}) \\ &\le \phi\Big(\max\Big\{d(Ax_{2n}, Bx_{2n-1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n-1}, Tx_{2n-1}), \\ &\qquad \frac{d(Tx_{2n-1}, Ax_{2n}) + d(Sx_{2n}, Bx_{2n-1})}{2}, \\ &\qquad \frac{d(Ax_{2n}, Tx_{2n-1}) + d(Ax_{2n}, Sx_{2n}) + d(Sx_{2n}, Bx_{2n-1})}{4[1 + d(Bx_{2n-1}, Tx_{2n-1})d(Sx_{2n}, Bx_{2n-1})d(Ax_{2n}, Tx_{2n-1})]}, \\ &\qquad \frac{d(Ax_{2n}, Tx_{2n-1}) + d(Sx_{2n}, Bx_{2n-1}) + d(Bx_{2n-1}, Tx_{2n-1})}{4[1 + d(Ax_{2n}, Tx_{2n-1})d(Ax_{2n}, Sx_{2n})d(Sx_{2n}, Bx_{2n-1})]}\Big\}\Big) \\ &= \phi\Big(\max\Big\{d(y_{2n}, y_{2n-1}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ &\qquad \frac{d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})}{2}, \\ &\qquad \frac{1}{4}\cdot\frac{d(y_{2n}, y_{2n}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n-1})}{1 + d(y_{2n-1}, y_{2n})d(y_{2n+1}, y_{2n-1})d(y_{2n}, y_{2n})}, \\ &\qquad \frac{1}{4}\cdot\frac{d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1}) + d(y_{2n-1}, y_{2n})}{1 + d(y_{2n}, y_{2n}))d(y_{2n}, y_{2n+1})d(y_{2n+1}, y_{2n-1})}\Big\}\Big). \end{aligned} \tag{2.5}$$

Since $\frac{a+b}{2} \le \max(a, b)$ for any $a, b \ge 0$, from the triangle inequality of d it follows that

$$d(y_{2n+1}, y_{2n+1}) + d(y_{2n}, y_{2n+1}) \le d(y_{2n-1}, y_{2n}) + 2d(y_{2n}, y_{2n+1})$$

so that

$$\begin{aligned} &\tfrac{1}{4}\left[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n+1})\right] \\ &\qquad \le \max\left\{\frac{1}{2}d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\right\}. \end{aligned} \tag{2.6}$$

Similarly,

$$\begin{aligned} &\tfrac{1}{4}\left[d(y_{2n+1}, y_{2n-1}) + d(y_{2n-1}, y_{2n})\right] \\ &\qquad \le \max\left\{d(y_{2n-1}, y_{2n}), \frac{1}{2}d(y_{2n}, y_{2n+1})\right\}. \end{aligned} \tag{2.7}$$

Using (2.6) and (2.7) in (2.5), we get

$$d(y_{2n+1}, y_{2n}) \le \phi\big(\max\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\}\big) \text{ for } n \ge 2. \tag{2.8}$$

Repeating the same argument, we get

$$d(y_{2n-1}, y_{2n}) \le \phi\big(\max\{d(y_{2n-2}, y_{2n-1}), d(y_{2n-1}, y_{2n})\}\big) \text{ for } n \ge 2. \tag{2.9}$$

Thus from (2.8) and (2.9), we write

$$d(y_n, y_{n+1}) \leq \phi\big(\max\{d(y_n, y_{n+1}), d(y_{n-1}, y_n)\}\big) \quad \text{for all} \quad n \geq 2. \tag{2.10}$$

If $d(y_m, y_{m+1}) > d(y_m, y_{m-1})$ for some $m \geq 2$, then $d(y_m, y_{m+1}) > 0$ so that (2.10) and the choice of ϕ would give a contradiction that $0 < d(y_m, y_{m+1}) \leq \phi(d(y_m, y_{m+1})) < d(y_m, y_{m+1})$. Therefore

$$d(y_n, y_{n+1}) \leq d(y_n, y_{n-1}) \quad \text{for all} \quad n \geq 2. \tag{2.11}$$

That is $\langle d(y_n, y_{n-1})\rangle_{n=1}^{\infty}$ is nonincreasing sequence of nonnegative numbers and hence converges to some $a \geq 0$. Using (2.11) in (2.10), we find that

$$d(y_n, y_{n+1}) \leq \phi\big(d(y_{n-1}, y_n)\big) \quad \text{for all} \quad n \geq 2.$$

Then employing the limit as $n \to \infty$ in this and using upper semicontinuity of ϕ, we get $a \leq \phi(a)$ so that $a = 0$, proving (2.4).

We claim that $\langle y_n\rangle_{n=1}^{\infty}$ is a Cauchy sequence. If possible we assume that our claim is false. Then for some $\epsilon > 0$, we choose sequences $\langle 2m_k\rangle_{k=1}^{\infty}$ and $\langle 2n_k\rangle_{k=1}^{\infty}$ of even integers such that $d(y_{2m_k}, y_{2n_k}) \geq \epsilon$ for $2m_k > 2n_k > k$ for all k. Let $2m_k$ be the smallest even integer with this property so that $d(y_{2m_k-2}, y_{2n_k}) \leq \epsilon$.

Using the triangle inequality of d and (2.4), above inequalities give

$$\begin{aligned}\lim_{n\to\infty} d(y_{2m_k}, y_{2n_k}) &= \lim_{n\to\infty} d(y_{2m_k}, y_{2n_k+1})\\ &= \lim_{n\to\infty} d(y_{2m_k+1}, y_{2n_k+1}) = \lim_{n\to\infty} d(y_{2m_k+1}, y_{2n_k+2}) = \epsilon.\end{aligned} \tag{2.12}$$

Now with $x = x_{2m_k}$ and $y = x_{2n_k+1}$, the inequality (2.2) gives

$$\begin{aligned} d(Sx_{2m_k}, Tx_{2n_k+1}) \leq \phi\Bigg(\max\Bigg\{ & d(Ax_{2m_k}, Bx_{2n_k+1}), d(Ax_{2m_k}, Sx_{2m_k}),\\ & d(Bx_{2n_k+1}, Tx_{2n_k+1}), \tfrac{d(Tx_{2n_k+1}, Ax_{2m_k}) + d(Sx_{2m_k}, Bx_{2n_k+1})}{2},\\ & \tfrac{1}{4}\cdot\tfrac{d(Ax_{2m_k}, Tx_{2n_k+1}) + d(Ax_{2m_k}, Sx_{2m_k}) + d(Sx_{2m_k}, Bx_{2n_k+1})}{1 + d(Bx_{2n_k+1}, Tx_{2n_k+1}) d(Sx_{2m_k}, Bx_{2n_k+1}) d(Ax_{2m_k}, Tx_{2n_k+1})},\\ & \tfrac{1}{4}\cdot\tfrac{d(Ax_{2m_k}, Tx_{2n_k+1}) + d(Sx_{2m_k}, Bx_{2n_k+1}) + d(Bx_{2n_k+1}, Tx_{2n_k+1})}{1 + d(Ax_{2m_k}, Tx_{2n_k+1}) d(Ax_{2m_k}, Sx_{2m_k}) d(Sx_{2m_k}, Bx_{2n_k+1})}\Bigg\}\Bigg)\end{aligned}$$

Again, using the triangle inequality, proceeding the limit as $k \to \infty$ in this, then using (2.4), (2.12) and the upper semicontinuity of ϕ, we get

$$0 < \epsilon \leq \phi\big(\max\{\epsilon, 0, 0, \tfrac{0+\epsilon+\epsilon}{2}, \tfrac{1}{4}\cdot\tfrac{\epsilon+0+0+\epsilon}{1+0}, \tfrac{1}{4}\cdot\tfrac{0+\epsilon+0+\epsilon}{1+0}\}\big) = \phi\big(\epsilon\big) < \epsilon.$$

This contradiction establishes that $\langle y_n\rangle_{n=1}^{\infty}$ must be a Cauchy sequence and its subsequences $\langle y_{2n}\rangle_{n=1}^{\infty}$ and $\langle y_{2n+1}\rangle_{n=1}^{\infty}$ are also Cauchy.

Case (i): $A(X)\bigcup B(X)$ is complete.
Then

$$\begin{aligned}\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} y_{2n+1}\\ = \lim_{n\to\infty} Bx_{2n+1} = z = Au = Bv \quad \text{for some} \quad u, v \in X.\end{aligned} \tag{2.13}$$

Hence $\langle y_n \rangle_{n=1}^{\infty}$ also converges to $z = Au = Bv$.
But then (2.2) with $x = u$ and $y = x_{2n+1}$ gives

$$\begin{aligned}d(Su, Tx_{2n+1}) \leq \phi\Bigg(\max \Bigg\{ & d(Au, Bx_{2n+1}), d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1}),\\ & \frac{1}{2}\left[d(Tx_{2n+1}, Au) + d(Su, Bx_{2n+1})\right],\\ & \frac{1}{4}\cdot\frac{d(Au, Tx_{2n+1}) + d(Au, Su) + d(Su, Bx_{2n+1})}{1 + d(Bx_{2n+1}, Tx_{2n+1})d(Su, Bx_{2n+1})d(Au, Tx_{2n+1})},\\ & \frac{1}{4}\cdot\frac{d(Au, Tx_{2n+1}) + d(Su, Bx_{2n+1}) + d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Au, Tx_{2n+1})d(Au, Su)d(Su, Bx_{2n+1})}\Bigg\}\Bigg).\end{aligned}$$

Since ϕ is usc, applying the limit as $n \to \infty$, this implies

$$\begin{aligned}d(Su, z) \leq & \phi\Bigg(\max \Bigg\{ 0, d(z, Su), 0, \frac{1}{2}d(Su, z),\\ & \frac{1}{4}\cdot\frac{0 + d(z, Su) + d(Su, z)}{1 + 0}, \frac{1}{4}\cdot\frac{0 + d(Su, z) + 0}{1 + 0}\Bigg\}\Bigg)\\ = & \phi\big(d(z, Su)\big)\end{aligned}$$

so that $Su = z$.

On the other hand, writing $x = x_{2n}$ and $y = v$ in (2.2) and simplifying, we get $Tv = z$. Thus

$$Su = Au = Tv = Bv = z, \tag{2.14}$$

which in view of weak compatibility of (A, S) and (B, T) implies that

$$Sz = Az = Tz = Bz. \tag{2.15}$$

Thus u is a common coincidence point for A, S, B and T.

Finally writing $x = u$ and $y = z$ in (2.2), we obtain

$$\begin{aligned}
d(z, Tz) =& d(Su, Tz) \\
\leq& \phi\Bigg(\max \Bigg\{ d(Au, Bz), d(Au, Su), d(Bz, Tz), \\
& \frac{1}{2}\left[d(Tz, Au) + d(Su, Bz)\right], \\
& \frac{1}{4} \cdot \frac{d(Au, Tz) + d(Au, Su) + d(Su, Bz}{1 + d(Bz, Tz)d(Su, Bz)d(Au, Tz)}, \\
& \frac{1}{4} \cdot \frac{d(Au, Tz) + d(Su, Bz) + d(Bz, Tz)}{1 + d(Au, Tz)d(Au, Su)d(Su, Bz)} \Bigg\}\Bigg) \\
=& \phi\Bigg(\max \Bigg\{ d(z, Tz), 0, 0, \frac{1}{2}\left[d(Tz, z) + d(z, Tz)\right], \\
& \frac{1}{4} \cdot \frac{d(z, Tz) + 0 + d(z, Tz}{1 + 0}, \frac{1}{4} \cdot \frac{d(z, Tz) + d(z, Tz) + 0}{1 + 0} \Bigg\}\Bigg) \\
=& \phi\big(d(z, Tz)\big)
\end{aligned}$$

so that $d(Tz, z) = 0$ or $Tz = z$. This again in view of (2.15) reveals that z is a common fixed point of A, S, B and T.

Case (ii): Let $S(X)$ be orbitally complete at x_0. Then $\langle y_n \rangle_{n=1}^{\infty}$ converges to $z \in S(X) \subset A(X)$. The conclusion follows from Case (i).

Case (iii): Let $T(X)$ be orbitally complete at x_0. Then $\langle y_n \rangle_{n=1}^{\infty}$ converges to $z \in T(X)$ and hence $z \in B(X)$, in view of (1.3). Again the conclusion follows from Case (i).

Uniqueness of the common fixed point follows directly from (2.2). ■

Setting $A = B$ in Theorem 2.1, we have

Corollary 2.2. Let A, S and T be self-maps on X satisfying (1.3), the condition (a) of Theorem 1.3 and the inequality

$$\begin{aligned}
d(Sx, Ty) \leq& \phi\Bigg(\max \Bigg\{ d(Ax, Ay), d(Ax, Sx), d(Ay, Ty), \frac{d(Ty, Ax) + d(Sx, ay)}{2}, \\
& \frac{1}{4} \cdot \frac{d(Ax, Ty) + d(Ax, Sx) + d(Sx, Ay)}{1 + d(Ay, Ty)d(Sx, Ay)d(Ax, Ty)}, \\
& \frac{1}{4} \cdot \frac{d(Ax, Ty) + d(Sx, Ay) + d(Ay, Ty)}{1 + d(Ax, Ty)d(Ax, Sx)d(Sx, Ay)} \Bigg\}\Bigg) \\
& \text{for all } x, y \in X,
\end{aligned} \tag{2.16}$$

where ϕ is usc contractive modulus. Then S, T, and A will have a common coincidence point. Further if either (A, S) or (A, T) is weakly compatible, then S, T and A will have a unique common fixed point.

Proof. Suppose that $A(X)$ is complete. Then as in Case (i) of Theorem 2.1, we see that $z = Au = Su$ We assume that (A, S) is weakly compatible. Then $ASu = SAu$ or $Az = Sz$. Then with $x = y = z$ in (2.16), we get

$$\begin{aligned} d(Sz, Tz) \leq & \phi\Bigg(\max \Bigg\{ d(Az, Az), d(Az, Sz), d(Az, Tz), \frac{d(Tz, Az) + d(Sz, Az)}{2}, \\ & \frac{1}{4} \cdot \frac{d(Az, Tz) + d(Az, Sz) + d(Sz, Az)}{1 + d(Az, Tz)d(Sz, Az)d(Az, Tz)}, \\ & \frac{1}{4} \cdot \frac{d(Az, Tz) + d(Sz, Az) + d(Az, Tz)}{1 + d(Az, Tz)d(Az, Sz)d(Sz, Az)} \Bigg\} \Bigg) \\ = & \phi\big(d(Sz, Tz)\big) \end{aligned} \tag{2.17}$$

so that $Sz = Tz$ and hence $Sz = Tz = Az$.

Similarly one can prove that z is a common coincidence point of A, S and T when (A, T) is weakly compatible. The remaining proof follows similar to that of Theorem 2.1. ■

If ϕ is nondecreasing, we see that the right hand side of (1.4) is less than or equal to the the right hand side of (2.16). That is, (2.16) is weaker than (1.4) whenever ϕ is nondcreasing. Thus Theorem 1.3 follows as a particular case of Corlooary 2.2, when ϕ is nondecreasing.

Remark 2.3. Corlooary 2.2 suggests us to conclude that weak compatibility of either pair is sufficient to obtain a common fixed point in case of three maps.

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