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Completely Prime PO Ideals and Prime PO Ideals in Ordered Semigroups

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ABSTRACT

In this paper the terms, completely prime ideal, prime ideal, completely semiprime ideal, semiprime ideal, prime radical and complete prime radical in a po semigroup are introduced. It is proved that in a po semigroup(i) A is a prime ideal of S,(ii) For *a*, *b* ∈ S; <*a*> <*b*> ⊆ A implies *a* ∈ A or *b* ∈ A,(iii) For *a*; *b* ∈ S; $S^1aS^1b\ S^1 \subseteq A$ implies *a* ∈ A or *b* ∈ A are equivalent. It is proved that a po ideal P of a po semigroup S is(1) completely prime iff S\P is either a po subsemigroup of S or empty(2) prime iff S\P is either an *m*-system or empty. It is also proved that every completely prime ideal of a po semigroup is prime. In a globally idempotent po semigroup, it is proved that every maximal ideal is prime. It is also proved that a globally idempotent po semigroup having a maximal ideal contains semisimple elements. It is proved that a po ideal A of a po semigroup S is completely semiprime if and only if *x* ∈ S, x^2 ∈ A implies *x* ∈ A. It is proved that if A is a completely semiprime ideal of a po semigroup S, then *x*, *y* ∈ S, *xy* ∈ A implies that *xy*S ⊆ A, *x*S*y* ⊆ A and S*xy* ⊆ A. It is also proved that every completely semiprime ideal of a po semigroup is semiprime. It is proved that a po ideal A of a po semigroup S is completely semiprime if and only if S\A is a *d*-system of S or empty. It is also proved that the nonempty intersection of a family of(1) completely prime ideals of a po semigroup is completely semiprime(2) prime ideals of a po semigroup is semiprime. And also proved that a po ideal Q of a semigroup S is(1) semiprime iff S\Q is either an *n*-system or empty. It is proved that if N is an *n*-system in a po semigroup S and *a* ∈N, then there exist an *m*-system M in S such that *a* ∈M and M ⊆N. It is proved that to each po ideal A of a po semigroup S, we associate four types of sets namely A_1, A_2, A_3, A_4 and we proved that $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$. In a commutative po semigroup, it is proved that $A_1 = A_2 = A_3 = A_4$ and in general po semigroups, it is proved that $A_1 \neq A_2 \neq A_3 \neq A_4$ by means of examples. It is proved that in a po semigroup S if A, B and C are ideals of S, then i) A⊆B⇒ $\sqrt{A} \subseteq \sqrt{B}$, ii) if A∩B ≠ ∅ then $\sqrt{AB} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$ and iii) $\sqrt{\sqrt{A}} = \sqrt{A}$. In a po semigroup S if A is a po ideal, then $\sqrt{A}$ is a semiprime ideal of S. It is proved that a po ideal Q of a po semigroup S is semiprime iff $\sqrt{Q} = Q$. It is proved that in a po semigroup S with identity there is a unique maximal ideal M such that $\sqrt{M^n} = M$ for all natural numbers *n*. Further it is proved that if A is a po ideal of a po semigroup S then $\sqrt{A}$ = { *x* ∈ S: every *m*-system of S containing *x* meets A } i.e., $\sqrt{A}$ = { $\{x \in S : M(x) \cap A \neq \emptyset\}$.

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KEY WORDS: completely prime ideal, prime ideal, completely semiprime ideal, semiprime ideal, prime radical and complete prime radical.

1. INTRODUCTION:

The algebraic theory of semigroups was widely studied by CLIFFORD [2], [3], PETRICH [5]. The ideal theory in general semigroups was developed by ANJANEYULU [1]. JIAN TANG and XIANG YUN XIE [4] studied on radicals of ideals of ordered semigroups. In this paper we introduce the notions of completely prime po ideal, prime po ideal, completely semiprime po ideal, semiprime po ideal, prime radical and complete prime radical and characterize completely prime po ideal, prime po ideal, completely semiprime po ideal, semiprime po ideal, prime radical and complete prime radical in po semigroups.

2. PRELIMINARIES:

DEFINITION 2.1: A semigroup S is said to be a ***partially ordered semigroup*** if S is a partially ordered set such that $a \le b \Rightarrow ax \le bx$, $xa \le xb$ for all $a, b, x \in$ S.

DEFINITION 2.2: A nonempty subset A of a po semigroup S is said to be ***po left ideal*** of S if i) b ∈ S, a ∈ A ⇒ ba ∈ A ii) a ∈ A and s ∈ S such that $s \le a \Rightarrow s \in$ A.

DEFINITION 2.3: A nonempty subset A of a po semigroup S is said to be ***po right ideal*** of S if i) b∈ S, a ∈ A ⇒ ab∈ A ii) a ∈ A and s ∈ S such that $s \le a \Rightarrow s \in$ A.

DEFINITION 2.4: A nonempty subset A of a po semigroup S is said to be ***po two sided ideal*** or ***po ideal*** of S if i) b ∈ S, a ∈ A ⇒ ba∈A, ab ∈ A ii) a ∈ A and s ∈ S such that $s \le a \Rightarrow s \in$ A.

THEOREM 2.5: Let S be a po semigroup and A ⊆ S, B ⊆ S. Then(i) A ⊆(A](ii)((A]] =(A](iii)(A](B] ⊆(AB](iv) A ⊆ B ⇒ A ⊆(B] and(v) A ⊆ B ⇒(A] ⊆(B].

THEOREM 2.6: The nonempty intersection of any family of po left ideals(or po right ideals or po ideals) of a po semigroup S is a po left ideal(or po right ideal or po ideal) of S.

3. COMPLETELY PRIME PO IDEALS AND PRIME PO IDEALS:

DEFINITION 3.1. A po(left/right) ideal A of a po semigroup S is said to be a *completely prime(left/right) ideal* of S provided $x, y \in$ S and $xy \in$ A implies either $x \in$ A or $y \in$ A.

THEOREM 3.2: A po ideal A of a po semigroup S is completely prime if and only if $x_1, x_2, \ldots, x_n \in$ S, n is a natural number, $x_1 x_2 \ldots x_n \in$ A $\Rightarrow x_i \in$ A for some i = 1, 2, 3, … n.

***Proof*:** Suppose that A is a completely prime po ideal of S.

Let $x_1, x_2, \ldots, x_n \in$ S where n is a natural number and $x_1 x_2 \ldots x_n \in$ A.

If $n = 1$ then clearly $x_1 \in$ A.

If $n = 2$ then $x_1x_2 \in$ A $\Rightarrow x_1 \in$ A or $x_2 \in$ A.

If $n = 3$ then $x_1x_2x_3 \in$ A $\Rightarrow x_1x_2 \in$ A or $x_3 \in$ A.

$\Rightarrow x_1 \in$ A or $x_2 \in$ A or $x_3 \in$ A.

Therefore by induction on n, $x_1 x_2 \ldots x_n \in$ A $\Rightarrow x_i \in$ A for some i = 1, 2, 3, …..n.

The converse part is trivial.

THEOREM 3.3: A po ideal A of a po semigroup S is completely prime if and only if S\A is either a subsemigroup of S or empty.

Proof: Suppose that A is a completely prime po ideal of S and S\A $\neq \emptyset$.

Let $a, b \in$ S\A. Then $a \notin$ A, $b \notin$ A.

Suppose if possible $ab \notin$ S\A.Then $ab \in$ A. Since A is completely prime, either $a \in$ A or $b \in$ A. It is a contradiction.

Therefore $ab \in$ S\A. Hence S\A is a subsemigroup of S.

Conversely suppose that S\A is a subsemigroup of S or S\A is empty.

If S\A is empty then A = S and hence A is completely prime.

Assume that S\A is a subsemigroup of S.

Let $a, b \in$ S and $ab \in$ A.

Suppose if possible $a \notin$ A and $b \notin$ A.

Then $a \in$ S\A and $b \in$ S\A.

Since S\A is a subsemigroup, $ab \in$ S\A and hence $ab \notin$ A. It is a contradiction.

Hence either $a \in$ A or $b \in$ A. Therefore A is a completely prime po ideal of S.

DEFINITION 3.4: A po ideal A of a po semigroup S is said to be a *prime ideal* of S provided X,Y are ideals of S and XY $\subseteq$ A $\Rightarrow$ X $\subseteq$ A or Y $\subseteq$ A.

THEOREM 3.5: In a po semigroup S, the following conditions are equivalent:

(i) A is a prime po ideal of S.

(ii) $a, b \in$ S; $\langle a \rangle \langle b \rangle \subseteq$ A implies $a \in$ A or $b \in$ A.

(iii) $a, b \in$ S; $S^1aS^1b\,S^1 \subseteq$ A implies $a \in$ A or $b \in$ A.

***Proof*:**(i) $\Rightarrow$(ii): Suppose that A is a prime po ideal of S. Then(i) $\Rightarrow$(ii) is obvious.

(ii) $\Rightarrow$(iii): Let $a, b \in$ S such that $S^1aS^1b\,S^1 \subseteq$ A.

Now $\langle a \rangle \langle b \rangle = (S^1aS^1)(S^1bS^1) \subseteq S^1aS^1b\,S^1 \subseteq$ A $\Rightarrow a \in$ A or $b \in$ A.

(iii) $\Rightarrow$(i): Suppose that $a, b \in$ S; $S^1aS^1bS^1 \subseteq$ A $\Rightarrow a \in$ A or $b \in$ A.

Let X, Y be the two ideals of S and XY $\subseteq$ A.

Suppose if possible X $\not\subseteq$ A, Y $\not\subseteq$ A.

Since X $\not\subseteq$ A, Y $\not\subseteq$ A, there exists $a, b \in$ S such that $a \in$ X and $a \notin$ A, $b \in$ Y and $b \notin$ A.

Now $S^1aS^1bS^1 \subseteq$ XY $\subseteq$ A $\Rightarrow a \in$A or $b \in$A. It is a contradiction. Therefore X $\subseteq$ A or Y $\subseteq$ A and hence A is a prime po ideal of S.

THEOREM 3.6: A po ideal A of a po semigroup S is prime if and only if $X_1, X_2, \ldots, X_n$ are ideals of S, n is a natural number, $X_1 X_2 \ldots\ldots X_n \subseteq A \Rightarrow X_i \subseteq A$ for some $i = 1, 2, 3, \ldots, n$.

***Proof*:** Suppose that A is a prime ideal of S.

Let $X_1, X_2, \ldots\ldots, X_n$ are ideals of S, n is a natural number and $X_1 X_2 \ldots\ldots X_n \subseteq A$

If $n = 1$ then clearly $X_1 \subseteq A$.

If $n = 2$ then $X_1X_2 \subseteq A \Rightarrow X_1 \subseteq A$ or $X_2 \subseteq A$.

If $n = 3$ then $X_1X_2X_3 \subseteq A \Rightarrow X_1X_2 \subseteq A$ or $X_3 \subseteq A$.

$\Rightarrow X_1 \subseteq A$ or $X_2 \subseteq A$ or $X_3 \subseteq A$.

Therefore by induction on n, $X_1 X_2 \ldots\ldots X_n \subseteq A \Rightarrow X_i \subseteq A$ for some $i = 1, 2, 3, \ldots, n$.

The converse part is trivial.

THEOREM 3.7: Every completely prime po ideal of a po semigroup S is a prime po ideal of S.

Proof: Suppose that A is a completely prime po ideal of a po semigroup S.

Let $a, b \in S$ and $<a> <b> \subseteq A$. Then $ab \in A$.

Since A is a completely prime po ideal of S, either $a \in A$ or $b \in A$.

Therefore A is a prime po ideal of S.

THEOREM 3.8: Let S be a commutative po semigroup. A po ideal A of S is a prime po ideal if and only if A is a completely prime po ideal.

DEFINITION 3.9: A nonempty subset A of a po semigroup S is said to be an ***m*-*system*** provided for any $a, b \in A$ implies that$(S^1aS^1bS^1) \cap A \neq \emptyset$.

THEOREM 3.10: A po ideal A of a po semigroup S is a prime po ideal of S if and only if S\A is an *m*-system of S or empty.

Proof: Suppose that A is a prime po ideal of a po semigroup S and $S \backslash A \neq \emptyset$.

Let $a, b \in S \backslash A$. Then $a \notin A$ and $b \notin A$.

Suppose if possible$(S^1aS^1bS^1) \cap S\backslash A = \emptyset$.

$(S^1aS^1bS^1) \cap S\backslash A = \emptyset \Rightarrow (S^1aS^1bS^1) \subseteq A$.

Since A is prime, either $a \in A$ or $b \in A$. It is a contradiction.

Therefore$(S^1aS^1bS^1) \cap S\backslash A \neq \emptyset$.

Hence S\A is an *m*-system.

Conversely suppose that S\A is either an *m*-system of S or $S\backslash A = \emptyset$.

If $S\backslash A = \emptyset$, then S = A and hence A is a prime po ideal of S.

Assume that S\A is an *m*-system of S.

Let $a, b \in S$ and $<a> <b> \subseteq A$.

Suppose if possible $a \notin A$ and $b \notin A$. Then $a, b \in S\backslash A$.

Since S\A is an *m*-system, then$(S^1aS^1bS^1) \cap S\backslash A \neq \emptyset$

$\Rightarrow (S^1aS^1bS^1) \nsubseteq A \Rightarrow <a><b> \nsubseteq A$. It is a contradiction.

Therefore $a \in A$ or $b \in A$. Hence A is a prime po ideal of S.

THEOREM 3.11: If S is a po semigroup such that $S = S^2$ then every maximal po ideal of S is a prime po ideal of S.

Proof: Let M be a maximal ideal of S. Let A, B be two ideals of S such that $AB \subseteq M$.

Suppose if possible $A \nsubseteq M$, $B \nsubseteq M$.

Now $A \nsubseteq M \Rightarrow M \cup A$ is a po ideal of S and $M \subset M \cup A \subseteq S$.

Since M is a maximal, $M \cup A = S$.

Similarly $B \nsubseteq M \Rightarrow M \cup B = S$.
Now $S = SS = (M \cup A)(M \cup B) \subseteq M \Rightarrow S \subseteq M$. Thus M = S. It is a contradiction.
Therefore either $A \subseteq M$ or $B \subseteq M$. Hence M is a prime po ideal of S.

THEOREM 3.12: If S is a po semigroup having maximal ideals and $S = S^2$ then S contains semisimple elements.

***Proof*:** Suppose that S is a globally idempotent po semigroup having maximal ideals. Let M be a maximal ideal of S. Then by theorem 3.11., M is prime.
Now if $a \in S \backslash M$ then $<a> \nsubseteq M$ and $<a>^2 \nsubseteq M$. Now $S = M \cup <a> = M \cup <a>^2$.
Therefore $a \in <a>^2$ and hence $<a> = <a>^2$. Thus a is a semisimple element.
Therefore S contains semisimple elements.

4. COMPLETELY SEMIPRIME PO IDEALS AND SEMIPRIME PO IDEALS:

DEFINITION 4.1: A po ideal A of a po semigroup S is said to be a *completely semiprime po ideal* provided $x \in S$, $x^n \in A$ for some natural number n implies $x \in A$.

THEOREM 4.2: A po ideal A of a po semigroup S is completely semiprime if and only if $x \in S$, $x^2 \in A$ implies $x \in A$.

Proof: Suppose that A is a completely semiprime po ideal of S.
Then clearly $x \in S$, $x^2 \in A \Rightarrow x \in A$.
Conversely suppose that $x \in S$, $x^2 \in A \Rightarrow x \in A$.
We prove that $x \in S$, $x^n \in A$, for some natural number n $\Rightarrow x \in A \longrightarrow (1)$, by induction on ***n*.**
Clearly(1) is true for $n = 2$.
Assume that(1) is true for $n = k$. i.e., $x^k \in A \Rightarrow x \in A$ for some natural number k.
Suppose that $x^{k+1} \in A$. Then $x^{k+1} \in A \Rightarrow x^{k+1}. x^{k-1} \in A \Rightarrow x^{2k} \in A \Rightarrow (x^k)^2 \in A \Rightarrow x^k \in A \Rightarrow x \in A$.
Therefore $x^k \in A \Rightarrow x \in A$.
By induction, $x^n \in A$ for some natural number n implies $x \in A$.
Therefore A is completely semiprime.

THEOREM 4.3: If A is a completely semiprime po ideal of a po semigroup S, then $x, y \in S$, $xy \in A$ implies that $xyS \subseteq A$, $xSy \subseteq A$ and $Sxy \subseteq A$.

***Proof*:** Let A be a completely semiprime po ideal of a semigroup S. Let $x, y \in S$, $xy \in$ A.
$(xy)^2 \in A$, A is completely semiprime implies $xy \in A$.
Let $s \in S$. Consider$(xys)^2 = (xys)(xys) = xys(xy)s(xy)sy \in A$.
$(xys)^2 \in A$, A is completely semiprime implies $xys \in A$.
Therefore $x, y \in S$, $xy \in A \Rightarrow xys \in A$ for all $s \in S \Rightarrow xyS \subseteq A$.
Now $xy \in A \Rightarrow (yx)^2 = (yx)(yx) = y(xy)(xy)x \in A$.
$(yx)^2 \in A$, A is completely semiprime $\Rightarrow yx \in A$.
Let $s \in S$. Consider$(xsy)^2 = (xsy)(xsy) = xs(yx)s(yx)sy \in A$.
$(xsy)^2 \in A$, A is completely semiprime implies $xsy \in A$.
Therefore $x, y \in S$, $xsy \in A$ for all $s \in S \Rightarrow xSy \subseteq A$.
If $s \in S$, then$(sxy)^2 = (sxy)(sxy) = s(xys)xy \in A$.
$(sxy)^2 \in A$, A is completely semiprime $\Rightarrow sxy \in A$.

Therefore $x, y \in$ S, $sxy \in$ A for all $s \in$ S $\Rightarrow$ S$xy \subseteq$ A.

COROLLARY 4.4: If a po ideal A of a po semigroup S is completely semiprime then $x, y \in$ S, $xy \in$ A $\Longrightarrow$ $\langle x\rangle \langle y\rangle \subseteq$ A.

THEOREM 4.5: Every completely prime po ideal of a po semigroup S is a completely semiprime po ideal of S.

Proof: Let A be a completely prime po ideal of a po semigroup S. Suppose that $x \in$ S and $x^2 \in$ A. Since A is a completely prime po ideal of S, $x \in$ A.

Therefore A is a completely semiprime po ideal.

THEOREM 4.6: Let A be a prime po ideal of a po semigroup S. If A is completely semiprime po ideal of S then A is completely prime.

***Proof*:** Let $x, y \in$ S and $xy \in$ A. Since A is completely semiprime, by corollary 4.4, $xy \in$ A $\Rightarrow$ $\langle x\rangle\langle y\rangle \subseteq$ A $\Rightarrow$ $x \in$ A or $y \in$ A. Hence A is completely prime.

THEOREM 4.7: The nonempty intersection of any family of a completely prime po ideal of a po semigroup S is a completely semiprime po ideal of S.

Proof: Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of a completely prime po ideals of S such that $\bigcap_{\alpha\in\Delta} A_\alpha \neq \varnothing$.

It is clear that $\bigcap_{\alpha\in\Delta} A_\alpha$ is a po ideal. Let $a \in$ S and $a^2 \in \bigcap_{\alpha\in\Delta} A_\alpha$. Then $a^2 \in A_\alpha$ for all $\alpha \in \Delta$.

Since A_α is completely prime, $a \in A_\alpha$ for all $\alpha \in \Delta$ and hence $a \in \bigcap_{\alpha\in\Delta} A_\alpha$.

Therefore $\bigcap_{\alpha\in\Delta} A_\alpha$ is a completely semiprime po ideal of S.

DEFINITION 4.8: Let S be a po semigroup. A nonempty subset A of S is said to be a *d*-system of S if $a \in$ A $\Rightarrow a^n \in$ A for all natural numbers *n*.

THEOREM 4.9: A po ideal A of a po semigroup S is completely semiprime if and only if S\A is a *d*-system of S or empty.

Proof: Suppose that A is a completely semiprime po ideal of S and S\A $\neq \varnothing$.

Let $a \in$ S\A. Then $a \notin$ A. Suppose if possible $a^n \notin$ S\A for some natural number *n*.

Then $a^n \in$ A. Since A is a completely semiprime po ideal then $a \in$ A. It is a contradiction.

Therefore $a^n \in$ S\A and hence S\A is a *d*-system.

Conversely suppose that S\A is a *d*-system of S or S\A is empty.

If S\A is empty then S = A and hence A is completely semiprime.

Assume that S\A is a *d*-system of S. Let $a \in$ S and $a^n \in$ A.

Suppose if possible $a \notin$ A. Then $a \in$ S\A.

Since S\A is a *d*-system, $a^n \in$ S\A. It is a contradiction. Then $a \in$ A.

Hence A is a completely semiprime po ideal of S.

DEFINITION 4.10: A po ideal A of a po semigroup S is said to be *semiprime po ideal* provided X is po ideal of S and $X^n \subseteq$ A for some natural number *n* implies X $\subseteq$ A.

THEOREM 4.11: A po ideal A of a po semigroup S is semiprime if and only if X is po ideal of S, $X^2 \subseteq A$ implies $X \subseteq A$.
***Proof*:** Suppose that A is a semiprime po ideal. Then clearly $X^2 \subseteq A \Rightarrow X \subseteq A$.
Conversely suppose that X is a po ideal of S, $X^2 \subseteq A \Rightarrow X \subseteq A$.
We prove that $X^n \subseteq A$, for some natural number $n \Rightarrow X \subseteq A \longrightarrow(1)$, by induction on n.
Since $X^2 \subseteq A$, then $X \subseteq A$,(1) is true for $n = 2$.
Assume that $X^k \subseteq A$ for some natural number k, $1 \leq k < n \Rightarrow X \subseteq A$.
Now $X^{k+1} \subseteq A \Rightarrow X^{k+1}.X^{k-1} \subseteq A \Rightarrow X^{2k} \subseteq A \Rightarrow (X^k)^2 \subseteq A \Rightarrow X^k \subseteq A \Rightarrow X \subseteq A$, by assumption. By induction $X^n \subseteq A$ for some natural number $n \Rightarrow X \subseteq A$.
Therefore A is semiprime.
THEOREM 4.12: Every prime po ideal of a po semigroup S is semiprime.
***Proof*:** Suppose that A is a prime po ideal of a po semigroup S. Let X be a po ideal of S such that $X^2 \subseteq A$. Since A is prime, $X \subseteq A$. Hence A is semiprime.
THEOREM 4.14: If A is a po ideal of a po semigroup S then the following are equivalent.

1. A is a semiprime ideal.
2. For $a \in S$; $<a>^2 \subseteq A$ implies $a \in A$.
3. For $a \in S$; $S^1aS^1a\,S^1 \subseteq A$ implies $a \in A$.

***Proof:*(i)** $\Rightarrow$(ii): Suppose that A is a semiprime ideal of S. Then(i) $\Rightarrow$(ii) is obvious.
(ii) $\Rightarrow$(iii): Let $a \in S$ such that $S^1a\,S^1a\,S^1 \subseteq A$.
Now $<a>^2 = (S^1a\,S^1)(S^1a\,S^1) \subseteq S^1a\,S^1a\,S^1 \subseteq A \Rightarrow a \in A$.
(iii) $\Rightarrow$(i): Suppose that $a \in S$; $S^1a\,S^1a\,S^1 \subseteq A \Rightarrow a \in A$.
Let X be a po ideal of S and $X^2 \subseteq A$.
Suppose if possible $X \not\subseteq A$.
Suppose $X \not\subseteq A$ there exists a such that $a \in X$ and $a \notin A$. $a \in X \Rightarrow a^2 \in X^2 \subseteq A$.
Now $S^1a\,S^1a\,S^1 \subseteq X^2 \subseteq A \Rightarrow a \in A$. It is a contradiction.
Therefore $X \subseteq A$ and hence A is a semiprime po ideal of S.
THEOREM 4.14: Every completely semiprime po ideal of a po semigroup S is a semiprime po ideal of S.
Proof: Suppose that A is a completely semiprime po ideal of a po semigroup S.
Let $a \in S$ and $<a>^n \subseteq A$ for some natural number n.
Now $aaa\ldots..a(n \text{ terms}) \in <a^{n>} \subseteq <a>^n \subseteq A \Rightarrow a^n \in A \Rightarrow a \in A \Rightarrow <a> \subseteq A$.
Therefore A is a semiprime po ideal of S.
THEOREM 4.15: Let S be a commutative po semigroup. A po ideal A of S is completely semiprime if and only if it is semiprime.
Proof: Suppose that A is a completely semiprime po ideal of S. By theorem 4.14, A is a semiprime po ideal of S.
Conversely suppose that A is a semiprime po ideal of S.
Let $x \in S$ and $x^n \in A$ for some natural number n.
Now $x^n \in A \Rightarrow <x>^n \subseteq A \Rightarrow <x> \subseteq A \Rightarrow x \in A$. Since A is semiprime.
Therefore A is a completely semiprime po ideal of S.
THEOREM 4.16: The nonempty intersection of any family of prime po ideals of a po semigroup S is a semiprime po ideal of S.

Proof: Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of prime ideals of S such that $\bigcap_{\alpha\in\Delta} A_\alpha \neq \emptyset$.

It is clear that $\bigcap_{\alpha\in\Delta} A_\alpha$ is a po ideal.

Let $a \in$ S, $<a>^2 \subseteq \bigcap_{\alpha\in\Delta} A_\alpha$ then $<a>^2 \subseteq A_\alpha$ for all $\alpha \in \Delta$.

Since A_α is a prime, $<a> \subseteq A_\alpha$ for all $\alpha \in \Delta$. So $<a> \in \bigcap_{\alpha\in\Delta} A_\alpha$.

Therefore $\bigcap_{\alpha\in\Delta} A_\alpha$ is a semiprime po ideal of S.

DEFINITION 4.17: A non-empty subset A of a po semigroup S is said to be an ***n*-system** provided $a \in$ A implies that($S^1a\, S^1a\, S^1$) ∩ A ≠ ∅.

THEOREM 4.18: Every *m*-system in a po semigroup S is an *n*-system.

Proof: Let A be an *m*-system of a po semigroup S. Let $a \in$ A. Since A is an *m*-system, $a \in$ A,($S^1a\, S^1a\, S^1$) ∩ A ≠ ∅. Therefore *A is* an *n*-system of S.

THEOREM 4.19: A po ideal A of a po semigroup S is a semiprime po ideal if and only if S\A is an *n*-system of S(or) empty.

Proof: Suppose that A is a semiprime po ideal of a po semigroup S and S\A $\neq \emptyset$.
Let $a \in$ S\A. Then $a \notin$ A.
Suppose if possible($S^1a\, S^1a\, S^1$) ∩ S\A = ∅.
($S^1a\, S^1a\, S^1$) ∩ S\A = ∅ ⇒($S^1a\, S^1a\, S^1$) ⊆ A.
Since A is semiprime, either $a \in$ A. It is a contradiction. Therefore($S^1a\, S^1a\, S^1$) ∩ S\A ≠ ∅.
Hence S\A is an *n*-system.
Conversely suppose that S\A is either an *n*-system or S\A = $\emptyset$.
If S\A = $\emptyset$ then S = A and hence A is a semiprime ideal.
Assume that S\A is an *n*-system of S. Let $a \in$ S and $<a>$ ⊆ A.
Let $a \in$ S\A, S\A is an *n*-system of S ⇒($S^1a\, S^1a\, S^1$) ∩ S\A ≠ ∅.
Suppose if possible $a \notin$ A. Then $a \in$ S\A. Since S\A is an *m*-system, Then($S^1a\, S^1a\, S^1$) ⊆ S\A ⇒($S^1a\, S^1a\, S^1$) ⊈ A ⇒ $<a>$ ⊈ A. It is a contradiction. Therefore $a \in$ A. Hence A is a semiprime po ideal of S.

THEOREM 4.20: If N is an *n*-system in a po semigroup S and $a \in$ N, then there exist an *m*-system M in S such that $a \in$ M and M $\subseteq$ N.

Proof: We construct a subset M of N as follows:
Define $a_1 = a$, Since $a_1 \in$ N and N is an *n*-system,($S^1a_1\, S^1a_1\, S^{1)}$ ∩ N ≠∅.

Let $a_2 \in$($S^1a_1\, S^1a_1\, S^{1)}$ ∩ N. Since $a_2 \in$ N and N is an *n*-system,($S^1a_2\, S^1a_2\, S^{1)}$ ∩ N $\neq \emptyset$ and so on.
In general, if a_i has been defined with $a_i \in$ N, choose a_{i+1} as an element of($S^1a_2\, S^1a_2\, S^{1)}$ ∩ N. Let M = $\{a_1, a_2 ..... a_{i,} a_{i+1} ....\}$. Now $a \in$ M and M $\subseteq$ N.
We now show that M is an *m*-system.

Let a_i, $a_j \in$ M(for $i \leq j$).
Then $a_{j+1} \in (S^1a_j\, S^1a_j\, S^1) \subseteq (S^1a_i\, S^1a_j\, S^1) \Rightarrow a_{j+1} = S^1a_i\, S^1a_j\, S^1$. But $a_{j+1} \in$ M, so $a_{j+1} \in (S^1a_i\, S^1a_j\, S^1) \cap$ M,
Therefore M is an *m*-system.

5. PRIME PO RADICAL AND COMPLETELY PRIME PO RADICAL:

NOTATION 5.1: If A is a po ideal of a po semigroup S, then we associate the following four types of sets.

A_1 = The intersection of all completely prime po ideals of S containing A.

$A_2 = \{x \in S: x^n \in A$ for some natural numbers $n\}$

A_3 = The intersection of all prime po ideals of S containing A.

$A_4 = \{x \in S: < x >^n \subseteq A$ for some natural number $n\}$

THEOREM 5.2: If A is a po ideal of a po semigroup S, then A $\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

Proof: i) A $\subseteq A_4$: Let $x \in$ A. Then $<x> \subseteq$ A and hence $x \in A_4$

Therefore A $\subseteq A_4$

ii) $A_4 \subseteq A_3$: Let $x \in A_4$. Then $< x >^n \subseteq$ A for some natural number *n*.
Let P be any prime po ideal of S containing A.
Then $< x >^n \subseteq$ A for some natural number $n \Rightarrow < x >^n \subseteq$ P.
Since P is prime, $<x> \subseteq$ P and hence $x \in$ P.
Since this is true for all prime ideals of P containing A, $x \in A_3$. Therefore $A_4 \subseteq A_3$

iii) $A_3 \subseteq A_2$: Let $x \in A_3$. Suppose if possible $x \notin A_2$.

Then $x^n \notin$ A for all natural number *n*.
Consider Q = $\bigcup x^n$ for all natural number *n*, and $x \in$ S.
Let *a*, $b \in$ Q. Then $a = (x)^r$, $b = (x)^s$ for some natural numbers *r, s*.
Therefore $ab = (x)^r\,(x)^s = x^{r+s} \in Q$ and hence Q is a subsemigroup of S.
By theorem 3.3, P = S\Q is a completely prime po ideal of S and $x \notin P$.
By theorem 3.8, P is a prime po ideal of S and $x \notin P$. Therefore $x \notin A_3$.
It is a contradiction. Therefore $x \in A_2$ and hence $A_3 \subseteq A_2$.

iv) $A_2 \subseteq A_1$: Let $x \in A_2$. Now x $\in A_2 \Rightarrow x^n \in$ A for some natural number *n*.
Let P be any completely prime po ideal of S containing A.
Then $x^n \in$ A $\subseteq$ P $\Rightarrow x^n \in$ P $\Rightarrow x \in$ P. Therefore $x \in A_1$. Therefore $A_2 \subseteq A_1$.

Hence $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

THEOREM 5.3: If A is a po ideal of a commutative po semigroup S, then $A_1 = A_2 = A_3 = A_4$

Proof: By theorem 5.2, A $\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.By theorem 3.8, in a commutative po semigroup S, a po ideal A is a prime po ideal if A is completely prime po ideal. So

$A_1 = A_3$. By theorem 4.15, in a commutative po semigroup S, a po ideal A is semiprime if and only if A is completely semiprime po ideal. So $A_4 = A_2$ .Hence $A_1 = A_2 = A_3 = A_4$.

NOTE 5.4: In an arbitrary po semigroup $A_1 \neq A_2 \neq A_3 \neq A_4$.

DEFINITION 5. 5: If A is a po ideal of a po semigroup S, then the intersection of all prime po ideals of S containing A is called *prime po radical* or simply *po radical* of A and it is denoted by $\sqrt{A}$ or *rad* A.

DEFINITION 5. 6: If A is a po ideal of a po semigroup S, then the intersection of all completely prime po ideals of S containing A is called *completely prime po radical* or simply *complete po radical* of A and it is denoted by *c.rad* A.

NOTE 5. 7: If A is a po ideal of a po semigroup S, then *rad* A = A_3, *c.rad* A = A_1 and *rad* A ⊆ *c.rad* A**.**

COROLLARY 5. 8: If $a \in \sqrt{A}$, then there exist a positive integer *n* such that $a^n \in$ A for some natural number $n \in$ N.

Proof: By theorem 5. 2, $A_3 \subseteq A_2$ and hence $a \in \sqrt{A} = A_3 \subseteq A_2$.

Therefore $a^n \in$ A for some natural number $n \in$ N.

COROLLARY 5. 9: If A is a po ideal of a commutative po semigroup S, then *rad* A = *c.rad* A.

proof: By theorem 5.3, *rad* A = *c.rad* A.

COROLLARY 5.10: If A is a po ideal of a po semigroup S then *c.rad* A is a completely semiprime po ideal of S.

proof: By theorem 4.5, *c.rad* A is a completely semiprime po ideal of S.

THEOREM 5.11: If A, B and C are any three ideals of a po ternary semigroup T, then

i) A $\subseteq$ B $\Rightarrow \sqrt{A} \subseteq \sqrt{B}$

ii) if A ∩ B ≠ ∅ then $\sqrt{AB} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$

iii) $\sqrt{\sqrt{A}} = \sqrt{A}$.

proof: i) Suppose that A$\subseteq$B. If P is a prime po ideal containing B then P is a prime po ideal containing A. Therefore $\sqrt{A} \subseteq \sqrt{B}$.

ii) Let P be a prime po ideal containing AB. Then AB ⊆ P $\Rightarrow$ A ⊆ P or B ⊆ P $\Rightarrow$ $A \cap B \subseteq P$. Therefore P is a prime po ideal containing $A \cap B$.

Therefore *rad(* $A \cap B$) ⊆ *rad(AB*).

Now let P be a prime po ideal containing $A \cap B$.

Then $A \cap B \subseteq P \Rightarrow AB \subseteq A \cap B \subseteq P \Rightarrow AB \subseteq P$.

Hence P is a prime po ideal containing *AB*. Therefore *rad(AB*) ⊆ *rad(* $A \cap B$).

Therefore *rad*(*AB*) = *rad(* $A \cap B$).

Since $A \cap B \neq \emptyset$, it is clear that $A \cap B$ is a po ideal in S. Let $x \in \sqrt{A \cap B}$.

Then there exists an natural number $n \in$ N such that $x^n \in A \cap B$.

Therefore $x^n \in$ A and $x^n \in$ B. It follows that $x \in \sqrt{A}$ and $x \in \sqrt{B}$. Therefore $x \in \sqrt{A} \cap \sqrt{B}$.

Consequently, $x \in \sqrt{A} \cap \sqrt{B}$ implies that there exists natural numbers *n*, $m \in$ N such that $x^n \in$ A and $x^m \in$ B. Clearly $x^{nm} \in$ A ∩ B.

Thus $x \in \sqrt{A \cap B}$. Therefore if A ∩ B ≠ ∅ then $\sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.

iii) $\sqrt{A}$ = The intersection of all prime po ideals of S containing A.

Now $\sqrt{\sqrt{A}}$ = The intersection of all prime po ideals of S containing $\sqrt{A}$.

= The intersection of all prime po ideals of S containing A = $\sqrt{A}$

Therefore $\sqrt{\sqrt{A}} = \sqrt{A}$.

THEOREM 5.12: If A is a po ideal of a po semigroup S then $\sqrt{A}$ is a semiprime po ideal of S.

proof: By theorem 4.16, $\sqrt{A}$ is a semiprime po ideal of S.

THEOREM 5.13: A po ideal Q of po semigroup S is a semiprime po ideal of S if and only if $\sqrt{Q} = Q$.

Proof: Suppose that Q is a semiprime po ideal. Clearly $Q \subseteq \sqrt{Q}$.

Suppose if possible $\sqrt{Q} \nsubseteq Q$.

Let $a \in \sqrt{Q}$ and $a \notin Q$. Now $a \notin Q \Rightarrow a \in S \setminus Q$ and Q is semiprime. By theorem 4.19, $S \setminus Q$ is an *n*-system. By theorem 4.20, there exists an *m*-system M such that $a \in M \subseteq S \setminus Q$.

$Q \subseteq S \setminus M$ and now $S \setminus M$ is a prime po ideal of S, $a \notin S \setminus M$. It is a contradiction.

Therefore $\sqrt{Q} \subseteq Q$. Hence $\sqrt{Q} = Q$.

Conversely suppose that Q is a po ideal of S such that $\sqrt{Q} = Q$.

By corollary 5.12, $\sqrt{Q}$ is a semiprime po ideal of S. Therefore Q is semiprime.

COROLLARY 5.14: A po ideal Q of a po semigroup S is a semiprime po ideal if and only if Q is the intersection of all prime po ideal of S contains Q.

***Proof*:** By theorem 5.13, Q is semiprime iff Q is the intersection of all prime po ideals of S contains Q.

COROLLARY 5.15: If A is a po ideal of a po semigroup S, then $\sqrt{A}$ is the smallest semiprime po ideal of S containing A.

Proof: We have that $\sqrt{A}$ is the intersection of all prime po ideals containing A in S. Since intersection of prime po ideals is semiprime, we have $\sqrt{A}$ is semiprime.

Further, let Q be any semiprime po ideal containing A, i.e. $A \subseteq Q$. So $\sqrt{A} \subseteq \sqrt{Q}$.

Since Q is semiprime, By theorem 5.13, $\sqrt{Q} = Q$. Therefore $\sqrt{A} \subseteq Q$.

Hence $\sqrt{A}$ is the smallest semiprime po ideal of S containing A.

THEOREM 5.16: If P is a prime ideal of a po semigroup S, then $\sqrt{(P)^n} = P$ for all natural numbers $n \in$ N.

Proof: We use induction on *n* to prove $\sqrt{P^n} = P$.

First we prove that $\sqrt{P} = P$. Since P is a prime ideal, $P \subseteq \sqrt{P} \subseteq P \Rightarrow \sqrt{P} = P$.

Assume that $\sqrt{P^k} = P$ for natural number *k* such that $1 \le k < n$.

Now $\sqrt{P^{k+1}} = \sqrt{P^k.P} = \sqrt{P^k} \cap \sqrt{P} = \sqrt{P} \cap \sqrt{P} = \sqrt{P} = P$.
Therefore $\sqrt{P^{k+2}} = P$. By induction $\sqrt{P^n}$ = P for all natural numbers $n \in$ N.

THEOREM 5.17: In a po semigroup S with identity there is a unique maximal ideal M such that $\sqrt{(M)^n}$ = M for all natural numbers $n \in$ N.

Proof: Since S contains identity, S is a globally idempotent po semigroup.
Since M is a maximal ideal of S, by theorem 3.11, M is prime.
By theorem 5.16, $\sqrt{(M)^n}$ = M for all natural numbers *n*.

Theorem 5.18: If A is a po ideal of a po semigroup S then $\sqrt{A}$ = { $x \in$ S: every *m*-system of S containing *x* meets A } i.e., $\sqrt{A} = \{x \in S : M(x) \cap A \neq \emptyset\}$.

Proof: Suppose that $x \in \sqrt{A}$. Let M be an m-system containing *x*.
Then S\M is a prime po ideal of S and $x \notin$ S\M. If M $\cap$ A = $\emptyset$ then A ⊆ S\M.
Since S\M is a prime po ideal containing A, $\sqrt{A}$ ⊆ S\M and hence $x \in$ S\M.
It is a contradiction. Therefore M(x) $\cap$ A $\neq \emptyset$. Hence $x \in \{x \in S : M(x) \cap A \neq \emptyset\}$.
Conversely suppose that $x \in \{x \in S : M(x) \cap A \neq \emptyset\}$.
Suppose if possible $x \notin \sqrt{A}$. Then there exists a prime po ideal P containing A such that $x \notin$ P. Now S\P is an *m*-system and $x \in$ S\P.
A ⊆ P $\Rightarrow$ S\P$\cap$A = $\emptyset \Rightarrow$ $x \notin \{x \in S : M(x) \cap A \neq \emptyset\}$. It is a contradiction. Therefore $x \in \sqrt{A}$. Thus $\sqrt{A} = \{x \in S : M(x) \cap A \neq \emptyset\}$.

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