Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 10, Number 4 (2014), pp. 557-567

http://www.ripublication.com

Universal Symbolic Expression for the Normalized Inner Product of the Position and Velocity Vectors of Conic Motion

M. A. Sharaf[1] and Z. A. Mominkhan[2]

[1]Department of Astronomy, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia Sharaf_adel@hotmail. com
[2]Department of Mathematics,
College of Science for Girls King Abdulaziz University, Jeddah, Saudi Arabia
Zammomin @hotmail. com

Abstract

In the present paper, universal symbolic expression of the normalized inner product of the position and velocity vectors σ of conic motion is developed in recursive power series forms. The importance of the quantity σ is due to its appearance in both, the initial and boundary value problems of space dynamics. Moreover, σ is related to the flight-path angle γ which is useful for specifying satellite's orientation or attitude. This orientation is crucial to determining the effective cross-sectional area required for both drag and solar-radiation perturbations. On the other hand, the importance of this analytical power series representation is that they are *invariant* under many operations because, addition, multiplication, exponent ion, integration, different ion, etc of a power series is also a power series. A fact which provides excellent flexibility in dealing with analytical as well as computational developments of the problems related to σ For computational developments, full recursive algorithm is developed for the coefficients of the series. Also an efficient method using continued fraction theory is provided for series evolution, moreover two devices are purposed to secure the convergence when the time interval $(t-t_0)$ is lager. In addition the algorithm dose not need the solution of Kepler's equation and its variants for parabolic and hyperbolic orbits.
Numerical applications of the algorithm are given for three orbits of different eccentricities, the results showed that it is accurate for any conic motion.

Keywords Space dynamics, orbit determination, symbolic expansions.

1. Introduction

It is undoubtly true that, the analytical formulae of space dynamics usually offer much deeper insight into the nature of the problems to which they refer. Moreover, the nowadays existing symbols used for manipulating digital computer programs, opened the gate towards establishing new branch of space dynamics known as the algorithmization of space dynamics [Brumberg, 1995]. A great effort has been devoted up to now, and is being devoted at present to develop symbolic computing algorithms for some problems of astrodynamics as well as astrophsics [e. g. Sharaf and Saad, 1997; Sharaf, et al, 1998;Sharaf, 2005, Sharaf**,** 2008, Sharaf and Sendi, 2011, Sharaf, et al, 2012, Sharaf and Saad, 2013].

In the absence of closed analytical solution of a given differential system the power series solution (which of course assumed to be convergent) can serve as the analytical representation of its solution. Moreover, it is worth noting that the power series is one of the most powerful methods of mathematical analysis and is some – times more convenient than the elementary functions especially when the problems are to be studied on computers. In fact, most computers often use series in the calculations of the majority of the elementary functions.

Coping with the above important line of recent approach, the present paper is devoted to establish universal symbolic expression for σ of conic motion.

The importance of the quantity σ is due to its appearance in both, the initial and boundary value problems of space dynamics. Moreover, σ is related to the flight-path angle γ which is useful for specifying satellite's orientation or attitude. This orientation is crucial to determining the effective cross-sectional area required for both drag and solar-radiation perturbations. On the other hand, the importance of these analytical power series representation is that, they are *invariant* under many operations because, addition, multiplication, exponent ion, integration, different ion, etc of a power series is also a power series. A fact which provides excellent flexibility in dealing with analytical as well as computational developments of the problems related to radial distance For computational developments, full recursive algorithm is developed for the coefficients of the series. Also an efficient method using continued fraction theory is provided for series evolution, moreover two devices are purposed to secure the convergence when the time interval $(t-t_0)$ is lager. In addition the algorithm dose not need the solution of Kepler's equation and its variants for parabolic and hyperbolic orbits.

2. Basic Formulations

2. 1 Differential equation

The polar equation of the relative motion of the two –body problem is given as

$$\ddot{r} - r\dot{\theta}^2 = -\mu / r^2 , \qquad (1)$$

where r is the radial distance, θ the true anomaly, μ is the gravitational parameter and $\dot{\theta} = \sqrt{\mu p}/r^2$, where p is the orbital parameter. Here a dot over a symbol denotes the derivative with respect to the time t. Since p is constant for the two body problem then Equation(1) could be written as

$$\ddot{q} = - \in q \,, \tag{2}$$

where

$$q = r - p \;, \tag{3}$$

$$\in = \frac{\mu}{r^3}.$$

2. 2. Lagrange's Fundamental Invariants

Lagrange's fundamental invariants [Battin, 1999] $\in$ **,** $\boldsymbol{\lambda}$ and $\boldsymbol{\psi}$ are defined as

$$\in = \frac{\mu}{r^3}.\,, \tag{4. 1}$$

$$\lambda = \frac{1}{r^2}\langle \mathbf{r}, \mathbf{v} \rangle \;, \tag{4. 2}$$

$$\psi = \frac{1}{r^2}\langle \mathbf{v}, \mathbf{v} \rangle \,, \tag{4-3}$$

where $\langle \mathbf{A}, \mathbf{B} \rangle$ is used to denote the scalar product of two the vectors $\mathbf{A}$ and $\mathbf{B}$. The quantities $\in$ **,** $\boldsymbol{\lambda}$ and $\boldsymbol{\psi}$ are "invariant" because they are independent of the selected

where $\langle \mathbf{A}, \mathbf{B} \rangle$ is used to denote the scalar product of two the vectors $\mathbf{A}$ and $\mathbf{B}$. The quantities $\in$ **,** $\boldsymbol{\lambda}$ and $\boldsymbol{\psi}$ are "invariant" because they are independent of the selected

coordinate system and "fundamental" because they form a closed set under the operation of time derivative, where

$$\frac{d \in}{dt} = -3 \in \lambda \;, \tag{5. 1}$$

$$\frac{d\lambda}{dt} = \psi - \in -2\lambda^2 \,, \tag{5. 2}$$

$$\frac{d\psi}{dt} = -2\lambda(\in + \psi). \tag{5. 3}$$

3. Solution by Power Series

3. 1 The basic differential equations

The basic differential equations that concerns us in the subsequent analysis are Equations (2) and (5) written as:

$$\frac{d^2 q}{dt^2} + \in q = 0, \tag{6. 1}$$

$$\frac{d\in}{dt}+3\in\lambda=0, \qquad (6.\ 2)$$

$$\frac{d\psi}{dt}+2\lambda(\in+\psi),=0 \qquad (6.\ 3)$$

$$\frac{d\lambda}{dt}+\in+2\lambda^2-\psi=0, \qquad (6.\ 4)$$

where $\in$ $,\lambda$ and ψ are defined by Equations (4).

3. 2 Power series solutions

Power series solutions for the above set of differential equations could be developed as follows Battin, [1999]

Expand each of the functions q, $\in$ $,\lambda$ and ψ in a Taylor's series in time we have

$$q=\sum_{n=0}^{\infty} q_n(t-t_0)^n, \qquad (7.\ 1)$$

$$\in=\sum_{n=0}^{\infty} \in_n(t-t_0)^n, \qquad (7.\ 2)$$

$$\lambda=\sum_{n=0}^{\infty} \lambda_n(t-t_0)^n, \qquad (7.\ 3)$$

$$\psi=\sum_{n=0}^{\infty} \psi_n(t-t_0)^n. \qquad (7\text{-}4)$$

The procedure is to substitute the four series given by Equations (7) into the four differential Equations (6) and then solve for the coefficients q_n $,\in_n$ $,\lambda_n$ and ψ_n by comparison of the coefficients of power of time. The central mathematical device used is the general relation

$$\left(\sum_{n=0}^{\infty}\alpha_n x^n\right)\left(\sum_{n=0}^{\infty}\beta_n x^n\right)=\sum_{n=0}^{\infty}\sum_{\nu=0}^{n}\alpha_\nu\,\beta_{n-\nu}\,x^n, \qquad (8)$$

which converts the product of two infinite series to a double summation.

3. 3 Recurrence relations

The resulting recurrence relations are

$$q_{n+2}=\frac{-1}{(n+1)(n+2)}\sum_{i=0}^{n}\in_i q_{n-i}, \qquad (9.\ 1)$$

$$\in_{n+1}=\frac{-3}{n+1}\sum_{i=0}^{n}\in_i\lambda_{n-i}, \qquad (9.\ 2)$$

$$\lambda_{n+1} = \frac{1}{n+1}\left\{\psi_n - \in_n - 2\sum_{i=0}^{n} \lambda_i \, \lambda_{n-i}\right\}, \tag{9. 3}$$

$$\psi_{n+1} = \frac{-2}{n+1} \sum_{i=0}^{n} \lambda_i \left(\in_{n-i} + \psi_{n-i}\right). \tag{9. 4}$$

3. 4. The starting values

The starting values for the recurrence relations of Equations (9) are $q_0 \equiv q(t_0)$, $q_1 = \dot{q}(t_0)$, $\in_0 \equiv \in(t_0)$, $\lambda_0 \equiv \lambda(t_0)$ and $\boldsymbol{\psi}_0 \equiv \boldsymbol{\psi}(\mathbf{t}_0)$ could be obtained from the known position and velocity vectors $\mathbf{r}_0(x_0, y_0, z_0)$ and $\dot{\mathbf{r}}_0(\dot{x}_0, \dot{y}_0, \dot{z}_0)$ at the time t_0 from the following algorithm

1 – $r_0 = \left(x_0^2 + y_0^2 + z_0^2\right)^{1/2}$;

2 - $\varepsilon_0 = \mu / r_0^3$;

3 – $q_1 = \left(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0\right)/r_0$;

4 - $\lambda_0 = q_1/r_0$;

5 – $\psi_0 = \left(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2\right)/r_0^2$;

6 - $h_x = y_0\dot{z}'_0 - z_0\,\dot{y}_0$;

7 – $h_y = z_0\dot{x}_0 - x_0\dot{z}_0$;

8 - $h_z = x_0\dot{y}_0 - y_0\dot{x}_0$;

9 – $q_0 = r_0 - \left(h_x^2 + h_y^2 + h_z^2\right)/\mu$;

10 - End

Having obtained the $q's, \in's, \lambda's$ and $\Psi's$ coefficients recursively from the Equations (9), then the power series expansion of r is valid for any conic orbits. (elliptic, parabolic, hyperbolic).

$$r = r_0 + \sum_{n=1}^{\infty} q_n \left(t - t_0\right)^n. \tag{10}$$

4. Symbolic and Numerical Applications

4. 1 Symbolic expansions

Using the symbolic manipulation capability of the software package *Mathematica*, we generate the coefficients q_j ; $j = 2,3,...,10$ in terms of the known initial values q_0, $q_1, \in_0$, λ_0 and ψ_0 and are listed in Table A of Appendix A..

4. 2 Numerical applications
The numerical evaluation of power series q (say):

$$q = \sum_{n=0}^{\infty} q_n (t - t_0)^n,$$

may diverges when $\Delta = (t - t_0)$ is large. To avoid this difficulty, the following two devices could be used:

4. 2. 1*Canonical units*
We uses the following transformations rules [Vallado, 1997]

Rules 1: Physical to Canonical

$\int \mu = 398600.4415 \text{km}^3 / \text{sec}^2 \to \mu = 1\,\text{ER}^2 / \text{TU}^3$

$\int$Distance p in km $\to$ Distance p^* in ER such that $p^* = p / \text{ER}$

$\int$Time t in sec $\to$ Time t^* in TU such that $t^* = t / f_1$

$\int$Speed s in km/sec $\to$ Speed s^* in ER/TU such that $s^* = s / f_2$

where, $\text{ER} = 6378.1363 \text{km}$ is the mean equatorial radius of the Earth., $f_1 = 806.8109 \text{sec}$. and $f_2 = 7.90536 \text{km/sec}$.

After performing the computations using these canonical units, we can convert the results into the physical units by applying the following rules 2

Rules 2: Canonical to Physical

$\int \mu = 1\,\text{ER}^2 / \text{TU}^3 \to \mu = 398600.4415 \text{km}^3 / \text{sec}^2$

$\int$Distance p^* in ER $\to$ Distance p in km such that $p = p^* \times \text{ER}$

$\int$Time t^* in TU $\to$ Time t in sec such that $t = t^* \times f_1$

$\int$ Speed s^* in ER/TU $\to$ Speed s in km/sec such that $s = s^* \times f_2$

If the time interval Δ were still large (because in orbit determination, Δ is usually small, but we include the device to cover all possibilities that may occurred)we can use the following device

4. 2. 2 Division of the time interval
In the case in which the time interval is large we can use the process of repeat decrementing of the time interval several times, so each time interval could be made as small as we desired.

4. 2. 3Continued fraction
In fact, continued fraction expansions are, generally far more efficient tools for

evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than the series. Due to the importance of accurate evaluations and the efficiency of continued fractions, We purpose to use them as the computational tools for evaluating the radial distance. To do so, two steps are to be performed:

1 Transform the given power series into continued fraction(poin a)

2 Evaluating the resulting continued fraction (point b)

a-Euler's transformation

Generally an infinite series (a power series is special case of it) of functions could be converted into a continued fraction according to Eulers transformation (Battin, 1999) which is

$$\sum_{k=0}^{\infty} U_K \equiv \cfrac{n_1}{d_1 + \cfrac{n_2}{d_2 + \cfrac{n_3}{d_3 + \cfrac{n_4}{\cdots}}}} \equiv \frac{n_1}{d_1 +}\,\frac{n_2}{d_2 +}\frac{n_3}{d_3 +}\,\frac{n_4}{d_4 +} + \ldots$$

where

$$n_1 = U_0\ ;\ n_2 = U_1;\ n_i = -U_{i-1} \times U_{i-3}, \forall i \geq 3$$

$$d_1 = 1\ ;\ d_j = U_{j-2} + U_{j-1} \forall i \geq 2.$$

b. Top-down continued fraction evaluation

There are several methods available for the evaluation of continued fraction. Traditionally, the fraction was either computed from the bottom up, or the numerator and denominator of the nth convergent were accumulated separately with three-term recurrence formulae. The draw back of the first method is, obviously, having to decide far down the fraction to being in order to ensure convergence. The draw back to the second method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm that works from top down while avoiding numerical difficulties would be ideal from a programming standpoint.

Gautschi (1967) proposed very concise algorithm to evaluate continued fraction from the top down and may be summarized as follows. If the continued fraction is written as

$$q = \frac{n_1}{d_1 +}\ \frac{n_2}{d_2 +}\ \frac{n_3}{d_3 +}\ \ldots$$

then initialize the following parameters

$$a_1 = 1\ ,\ \ b_1 = n_1 / d_1\ ,\ q_1 = n_1 / d_1$$

and iterate (k=1, 2, …) according to

$$a_{k+1} = \frac{1}{1 + \frac{n_{k+1}}{d_k d_{k+1}} a_k}$$

$$b_{k+1} = (a_k - 1)b_k,$$

$$q_{k+1} = q_k + b_{k+1}.$$

In the limit, the q sequence converges to the value of the continued fraction

4. 2. 4Numerical examples

In what follows, we shall consider three orbits the first is elliptic, the second is parabolic, while the third is hyperbolic

The initial position and velocity vectors of the orbits are listed in Tables I and II

Table I: The initial position vector

Orbit	x_0 (km)	y_0 (km)	z_0 (km)
1	5096. 530625	3997. 328251	-1767. 35171
2	-1616. 940994	7756. 699643	-7712. 188395
3	10000	0. 0	0. 0

Table II: The initial velocity vector

Orbit	$\dot{x}_0$ (km/sec)	$\dot{y}_0$ (km/sec)	$\dot{z}_0$ (km/sec)
1	4. 683016085	0. 602386847	4. 217758697
2	-0. 6730303137	8. 434930957	0. 7055483746
3	0. 0	0. 0	9. 2

For all three orbits we take $t_0 = 0$, $t = t = 500\,\text{sec.}, m = 10$, where m is the number of terms in the series of Equation (10). Applying the recurrent computations of Section 3together with Rules 1 of Section 4. 2. 1 we get respectively for the value of q_0 of the three orbits the values $q_{E0} = 0.481275\text{ER}$, $q_{P0} = -0.266446\text{ER}$ and $q_{H0} = 0.481275\text{ER}$. The other coefficients $q_j; j = 1,2,\cdots,m$ are listed in the first three columns of Table III together with the value Δ^j in the fourth column. Finally we compare the value of the radial distance r at $t = 500\,\text{sec} \equiv 0.619724\,\text{TU}$ as computed from the above algorithm with its exact value, these comparisons are listed in Table IV for the three orbits

Table III: The values of the q's coefficients & Δ^j for the three orbits

j	q_{Ej}	q_{Pj}	q_{Hj}	Δ^j
1	0.354604	0.698714	0.	0.619724
2	˜ 0.206308	0.0255706	0.228516	0.384058
3	0.0188294	˜ 0.0326577	0.	0.23801
4	˜ 0.00328605	0.0170693	˜ 0.0215942	0.1475
5	˜ 0.0036932	˜ 0.00697602	0.	0.0914095
6	0.00443825	0.00219	0.00362221	0.0566486
7	˜ 0.00398278	˜ 0.000350869	0.	0.0351065
8	0.00313643	˜ 0.000167967	˜ 0.000745091	0.0217563
9	˜ 0.0023212	0.00020834	0.	0.0134829
10	0.00164359	˜ 0.000131368	0.000170544	0.00835569

Table IV: Comparison between the computed and exact values of r

Orbit	Computed r (ER)	Exact r(ER)
1	1. 1969909	
2	2. 17063610	
3	1. 6526248	

Although we used relatively small number of terms for the power series, Table IV shows that the present algorithm is accurate enough ($\approx O(10^{-6})$) for predicating radial distance of any conic orbit.

In concluded the present paper, we stress that, universal symbolic expression for radial distance of conic motion in recursive power series forms is developed. The importance of this analytical power series representation is that, they are *invariant* under many operations because, addition, multiplication, exponent ion, integration, different ion, etc of a power series is also a power series. A fact which provides excellent flexibility in dealing with analytical as well as computational developments of the problems related to radial distance. For computational developments, full recursive algorithm is developed for the coefficients of the series. Also an efficient method using continued fraction theory is provided for series evolution, moreover two devices are purposed to secure the convergence when the time interval $(t - t_0)$ is lager. In addition dose not need the solution of Kepler's equation and its variants for parabolic and hyperbolic orbits.

Numerical applications of the algorithm are given for three orbits of different eccentricities, the results showed that it is accurate for any conic motion.

Appendix A

Table A: Symbolic expressions of the q' coefficients

$q_2' = \frac{1}{2}\, q_0\, \sigma_0$

$q_3' = \frac{1}{6}\, \sigma_0\, (q_1 \ldots 3 q_0\, \sigma_0)$

$q_4' = \frac{1}{24}\, \sigma_0\, (6 q_1 \ldots q_0 \ldots 2 \ldots 3 \ldots 5 \ldots^2 \ldots \sigma_0)$

$q_5' = \frac{1}{120}\, \sigma_0\, (15 q_0 \ldots 2 \ldots 7 \ldots^2 \; 3 \ldots q_1 \ldots 8 \ldots 9 \ldots 5 \ldots^2 \ldots)$

$q_6' = \frac{1}{720}\, \sigma_0\, (30 q_1 \ldots 5 \ldots 14 \ldots 6 \ldots q_0 \ldots 22 \ldots^2 \; 6 \ldots 20 \ldots^2 \; 11 \ldots$
$45 \; 21 \ldots 14 \ldots^2 \ldots)$

$q_7' = \frac{1}{5040}\, (\ldots q_1 \; 172 \ldots 36 \ldots 20 \ldots 11 \ldots 225 \; 21 \ldots 14 \ldots^2 \ldots$
$63 q_0 \; 12 \ldots^2 \; 4 \ldots 25 \ldots 9 \ldots 5 \; 33 \ldots 30 \ldots^2 \; 5 \ldots^2)$

$q_8' = \frac{1}{40320}\, (\ldots 126 q_1 \ldots 52 \ldots^2 \; 14 \ldots 25 \ldots 9 \ldots 15 \; 33 \ldots 30 \ldots^2 \; 5 \ldots$
$q_0 \; 584 \ldots^3 \; 36 \ldots^2 \; 560 \ldots 73 \ldots 54 \ldots 1925 \ldots 1120 \ldots^2 \; 67 \ldots^2$
$315 \; 429 \ldots 495 \ldots^4 \; 135 \ldots^2 \ldots 5 \ldots^3)$

$q_9' = \frac{1}{362880}\, (\ldots 15 q_0 \ldots 2368 \ldots^3 \; 444 \ldots^2 \; 27 \ldots 24 \ldots$
$18 \ldots 2007 \ldots 5698 \ldots^2 \; 827 \ldots 189 \; 715 \ldots 1001 \ldots^4 \; 385 \ldots^2 \; 35 \ldots^3$
$q_1 \; 2136 \ldots^3 \; 108 \ldots^2 \; 1785 \ldots 232 \ldots 432 \ldots 1925 \ldots 1120 \ldots^2 \; 67 \ldots^2$
$2205 \; 429 \ldots 495 \ldots^4 \; 135 \ldots^2 \ldots 5 \ldots^3)$

$q_{10}' = \frac{1}{3628800}\, (\ldots 30 q_1 \ldots 15220 \ldots^3 \; 12 \ldots^2 \; 14938 \ldots 4647 \ldots$
$81 \ldots 2007 \ldots 5698 \ldots^2 \; 827 \ldots 756 \; 715 \ldots 1001 \ldots^4 \; 385 \ldots^2 \; 35 \ldots^3$
$q_0 \; 28384 \ldots^4 \; 48 \ldots^3 \; 31735 \ldots 3548 \ldots 54 \ldots^2 \; 245245 \ldots 126940 \ldots^2 \; 6559 \ldots^2$
$90 \ldots 420420 \ldots^6 \; 441441 \ldots^4 \; 107514 \ldots^2 \; 3461 \ldots^3$
$14175 \; 2431 \ldots 4004 \ldots^6 \; 2002 \ldots^4 \; 308 \ldots^2 \ldots^3 \; 7 \ldots^4)$

References

[1] Battin, R. H. (1999), *An Introduction to the Mathematics and Methods of Astrodynamics*, Revised Edition, AIAA, Education series, New York.

[2] Brumberg, V. A. (1995), *Analytical Techniques of Celestial Mechanics*, Springer-Verlag, Berlin, Heidelberg.

[3] Sharaf, M. A. and Saad, A. S. (1997), 'Analytical Expansion of the Earth's Zonal Potential in Terms of Ks Regular Elements, Celestial Mechanics and Dynamical Astronomy 66, 181.

[4] Sharaf, M. A. ; Saad, A. S. and Sharaf, A. A. (1998), Unified Symbolic Algorithm of Gauss Method for Near-Parabolic Orbits, Celestial Mechanics and Dynamical Astronomy, 70, 201.

[5] Vallado, A. D. (1997), *Fundamentals of Astrodynamics and Applications*, McGraw-Hill Companies, Inc. USA.

[6] Sharaf, M. A: 2005, Symbolic Approach for the Applicability of the Third. Integral of Motion. Bulletin of Pure and Applied Sciences. Vol. 24D(No. 2), 431-442

[7] Sharaf, M. A: 2008, " Symbolic Analytical Developments of the Zero Pressure Cosmological Model of the Universe" Astrophysics and Space Science 14, 211-223

[8] Sharaf, M. A. and Sendi, A. M.: 2011, Analytical solution for Density in Globular Clusters, Journal of Astrophysics and Astronomy, 32, 371-376

[9] Sharaf, M. A, Hassan, I. A., Ghoneim, R and Alshaery, A. A: 2012"Symbolic Solution of the Three Dimensional Restricted Three-Body Problem" Al-Azhar Bull. Sci. Vol. 23, No. 1(June): PP17-28

[10] Sharaf, M. A. and Saad, A. S., 2013, Symbolic and Graphical Computations of a Class of Slightly Perturbed Equations, *Applied Mathematics, vol. 4, no. 4, pp. 817-824.*

[11] Gautschi, W.: 1967" Computational Aspects of Three-term Recurrence Relations", SIAM Review, Vol. 9, No. 1, January.