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Quantum Algorithm for Minimum *K*-clustering Problem by Numbering Method

Toru Fujimura

*Department of Chemistry, Industrial Property Cooperation Center,
1-2-15, Kiba, Koto-ku, Tokyo 135-0042, Japan
E-mail: tfujimura8@gmail.com*

Abstract

A quantum algorithm for the minimum *K*-clustering problem by a numbering method and its example are reported. When *n* points are parted by *K* subsets, it is decided whether the distance of each pair in the same subset is a natural number *M* or less or not for all of subsets. A computational complexity of a classical computation is K^n. The computational complexity becomes about n^2 by the quantum algorithm that uses quantum phase inversion gates, quantum inversion about mean gates and the numbering method. Therefore, a polynomial time process becomes possible.

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Keywords: Quantum algorithm, minimum K-clustering problem, numbering method, computational complexity, polynomial time.

1. Introduction

Haroche and Wineland [1] developed methods for measuring and manipulating individual quantum particles, which were the very first steps towards building a quantum computer. The algorithms of the quantum computer by Deutsch-Jozsa [2−4], Shor [3−5], Grover [3, 6, 7] and so on are known. A quantum algorithm for the 3-SAT problem by a numbering method has recently been reported by Fujimura [8]. Its computational complexity becomes a polynomial time. The minimum *K*-clustering problem [9] is examined by the numbering method this time. Therefore, its result is reported.

2. Minimum *K*-clustering Problem

When n points are parted by K subsets, it is searched whether the distance of each pair in the same subset is a natural number M or less or not for all of subsets [9].

3. Quantum Algorithm

It is assumed that n points are parted by K subsets, each point is x((point number), (subset number)) = $x(f, h_f)$ [$1 \leq f \leq n$. $1 \leq h_f \leq K$. f and h_f are integers.] and a natural number is M. When a combination $(x(1, h_1), x(2, h_2), \cdots, x(n-1, h_{n-1}), x(n, h_n))$ is labeled $(h_1 - 1, h_2 - 1, \cdots, h_{n-1} - 1, h_n - 1)$, $V(N)$ is the N-th $(h_1 - 1, h_2 - 1, \cdots, h_{n-1} - 1, h_n - 1)$ [$0 \leq N \leq K^n - 1$. N is an integer. $V(0)$ is $(0, 0, \cdots, 0, 0)$. $V(K^n - 1)$ is $(K - 1, K - 1, \cdots, K - 1, K - 1)$.]. This method is named the numbering method for this problem. g is the minimum integer that follows $K^n/K! \leq 4^g = 2^{2g}$, because a number of combinations of an answer is at least $K!$.

First of all, quantum registers $|a_1>, |a_2>, \cdots, |a_n>, |b_1>, |b_2>, |c_1>, |c_2>, \cdots, |c_K>, |d>, |e_1>, |e_2>$ and $|e_3>$ are prepared. When P is the minimum integer that is $log_2 K$ or more, each of $|a_f>$ that f is an integer from 1 to n is consisted of P quantum bits [= qubits]. States of $|a_f>, |b_1>, |b_2>, |c_i>$ [$1 \leq i \leq K$. i is an integer.], $|d>, |e_1>, |e_2>$ and $|e_3>$ are $a_f, b_1, b_2, c_i, d, e_1, e_2$ and e_3, respectively.

- Step 1: Each qubit of $|a_f>, |b_1>, |b_2>, |c_i>, |d>, |e_1>, |e_2>$ and $|e_3>$ is set $|0>$.
- Step 2: The Hadamard gate $\boxed{\text{H}}$ [3, 4] acts on each qubit of $|a_f>$. It changes them for entangled states. The total states are $(2^P)^n$.
- Step 3: It is assumed that a quantum gate (A) changes $|b_1>$ for $|1>$ in $a_f < K$, or it changes $|b_1>$ for $|0>$ in the others of a_f, and it changes $|b_2>$ for $|a_1, a_2, \cdots, a_{f-1}, a_f>$ at $|a_f>$. As a target state for $|b_1>$ is 1, quantum phase inversion gates (PI) and quantum inversion about mean gates (IM) [3, 6, 7] act on $|b_1>$. When Q is the minimum even integer that is $(2^P/K)^{1/2}$ or more, the total number that (PI) and (IM) act on $|b_1>$ is Q because they are a couple. Next, an observation gate (OB) observes $|b_1>$. These actions are repeated sequentially from $|a_1>$ to $|a_n>$. Therefore, each state of $|a_f>$ is $0, 1, \cdots, K-2$ or $K-1$, and the total states become K^n [$=W_0$].
- Step 4: It is assumed that a quantum gate (B) changes $|c_{y+1}>$ for $|c_{y+1} + 1>$ at $a_p = a_q = y$ and $|x(p, y+1) - x(q, y+1)| \leq M$ [$1 \leq p < q \leq n$. $0 \leq y \leq K - 1$. p, q and y are integers.], or it changes $|d>$ for $|d + 1>$ at $a_p = a_q = y$ and $|x(p, y+1) - x(q, y+1)| > M$. These actions are repeated sequentially from $p = 1$ and $q = 2$ to $p = n - 1$ and $q = n$.
- Step 5: It is assumed that a quantum gate (C) changes $|e_1>$ for $|e_1 + 1>$ in $c_i > 0$. This action is repeated sequentially from $i = 1$ to $i = K$. And then (C) changes $|e_1>$ for $|e_1 + 1>$ at d = 0.
- Step 6: It is assumed that a quantum gate (D) changes $|e_2>$ for $|e_2> = |V(K^n - 1)>$ at $e_1 = K + 1$, or it changes $|e_2>$ for $|e_2 + b_2> = |a_1, a_2, \cdots, a_{n-1}, a_n>$ in the others of e_1.
- Step 7: It is assumed that a quantum gate (E_1) changes $|e_3>$ for $|1>$ in $V(0) \leq e_2 \leq V((W_0/4) - (K! + 1))$ or $e_2 = V(K^n - 1)$, or it changes $|e_3>$ for $|0>$ in the others of e_2. As the target state for $|e_3>$ is 1, (PI) and (IM) act on $|e_3>$. The number of

the data that is included in $V(0) \leq e_2 \leq V((W_0/4) - (K! + 1))$ or $e_2 = V(K^n - 1)$ is $W_1 \approx W_0/4$. When R_1 is the minimum even integer that is $(W_0/W_1)^{1/2} \approx (W_0/(W_0/4))^{1/2}$ or more, the total number that (*PI*) and (*IM*) act on $|e_3>$ is $R_1 \approx 2$. Next, (*OB*) observes $|e_3>$, and the data of W_1 remain. Similarly, (E_z) [$2 \leq z \leq g - 1$. z is an integer.] changes $|e_3>$ for $|1>$ in $V(0) \leq e_2 \leq V((W_0/4^z) - (K! + 1))$ or $e_2 = V(K^n - 1)$, or it changes $|e_3>$ for $|0>$ in the others of e_2. As the target state for $|e_3>$ is 1, (*PI*) and (*IM*) act on $|e_3>$. The number of the data that is included in $V(0) \leq e_2 \leq V((W_0/4^z) - (K! + 1))$ or $e_2 = V(K^n - 1)$ is $W_z \approx W_0/4^z$. When R_z is the minimum even integer that is $(W_{z-1}/W_z)^{1/2} \approx ((W_0/4^{z-1})/(W_0/4^z))^{1/2}$ or more, the total number that (*PI*) and (*IM*) act on $|e_3>$ is $R_z \approx 2$. Next, (*OB*) observes $|e_3>$, and the data of W_z remain. These actions are repeated sequentially from 2 to $g - 1$ at z. (E_g) changes $|e_3>$ for $|1>$ at $e_2 = V(K^n - 1)$, or it changes $|e_3>$ for $|0>$ in the others of e_2. As the target state for $|e_3>$ is 1, (*PI*) and (*IM*) act on $|e_3>$. The number of the data that is included at $e_2 = V(K^n - 1)$ is $W_g \approx K! \approx W_0/4^g$. When R_g is the minimum even integer that is $(W_{g-1}/W_g)^{1/2} \approx ((W_0/4^{g-1})/(W_0/4^g))^{1/2}$ or more, the total number that (*PI*) and (*IM*) act on $|e_3>$ is $R_g \approx 2$. Next, (*OB*) observes $|a_f>$, $|b_1>$, $|b_2>$, $|c_i>$, $|d>$, $|e_1>$ $|e_2>$ and $|e_3>$, and one of the data of W_g remains. Therefore, one example of combinations that are $|x(p, y + 1) - x(q, y + 1)| \leq M$ is obtained.

4. Numerical Computation

It is assumed that there are $1 \leq f \leq n = 6$, $1 \leq h_f \leq K = 3$, $1 \leq i \leq K = 3$, $x(f, h_f)$ [$x(1, h_1) = 1$, $x(2, h_2) = 2$, $x(3, h_3) = 3$, $x(4, h_4) = 4$, $x(5, h_5) = 5$, $x(6, h_6) = 6$], $M = 3$, $g = 4$, $V(728) = (2, 2, 2, 2, 2, 2)$, $V(175) = (0, 2, 0, 1, 1, 1)$, $V(39) = (0, 0, 1, 1, 1, 0)$ and $V(4) = (0, 0, 0, 0, 1, 1)$.

First of all, $|a_f>$, $|b_1>$, $|b_2>$, $|c_i>$, $|d>$, $|e_1>$, $|e_2>$ and $|e_3>$ are prepared. When P is the minimum integer that is $log_2 K = log_2 3 \approx 1.6 \leq 2 = P$, each of $|a_f>$ that f is the integer from 1 to 6 is consisted of $P = 2$ qubits. States of $|a_f>$, $|b_1>$, $|b_2>$, $|c_i>$, $|d>$, $|e_1>$, $|e_2>$ and $|e_3>$ are a_f, b_1, b_2, c_i, d, e_1, e_2 and e_3, respectively.

- Step 1: Each qubit of $|a_f>$, $|b_1>$, $|b_2>$, $|c_i>$, $|d>$, $|e_1>$, $|e_2>$ and $|e_3>$ is set $|0>$.
- Step 2: $\boxed{H}$ acts on each qubit of $|a_f>$. It changes them for entangled states. The total states are $(2^P)^n = (2^2)^6$.
- Step 3: (*A*) changes $|b_1>$ for $|1>$ in $a_f < 3$, or it changes $|b_1>$ for $|0>$ in the others of a_f, and it changes $|b_2>$ for $|a_1, a_2, \cdots, a_{f-1}, a_f>$ at $|a_f>$. As the target state for $|b_1>$ is 1, (*PI*) and (*IM*) act on $|b_1>$. When Q is the minimum even integer that is $(2^2/3)^{1/2} \approx 1.2 \leq 2 = Q$, the total number that (*PI*) and (*IM*) act on $|b_1>$ is $Q \approx 2$. Next, (*OB*) observes $|b_1>$. These actions are repeated sequentially from $|a_1>$ to $|a_6>$. Therefore, each state of $|a_f>$ is 0, 1 or 2, and the total states become 3^6 [$= W_0$].
- Step 4: (*B*) changes $|c_{y+1}>$ for $|c_{y+1} + 1>$ at $a_p = a_q = y$ and $|x(p, y + 1) - x(q, y + 1)| \leq M$ [$1 \leq p < q \leq 6$. $0 \leq y \leq 2$. p, q and y are integers.], or it changes $|d>$ for $|d + 1>$ at $a_p = a_q = y$ and $|x(p, y + 1) - x(q, y + 1)| > M$. These actions are repeated sequentially from $p = 1$ and $q = 2$ to $p = 5$ and $q = 6$.
- Step 5: (*C*) changes $|e_1>$ for $|e_1 + 1>$ in $c_i > 0$. This action is repeated

sequentially from $i = 1$ to $i = 3$. And then (C) changes $|e_1>$ for $|e_1 + 1>$ at d = 0.

- Step 6: (D) changes $|e_2>$ for $|e_2> = |V(728)>$ at $e_1 = 4$, or it changes $|e_2>$ for $|e_2> = |b_2> = |a_1, a_2, a_3, a_4, a_5, a_6>$ in the others of e_1.
- Step 7: (E_1) changes $|e_3>$ for $|1>$ in $V(0) = (0, 0, 0, 0, 0, 0) \leq e_2 \leq V((3^6/4) - (3! + 1)) \approx V(175) = (0, 2, 0, 1, 1, 1)$ or $e_2 = V(728)$, or it changes $|e_3>$ for $|0>$ in the others of e_2. As the target state for $|e_3>$ is 1, (PI) and (IM) act on $|e_3>$. The number of the data that is included in $V(0) \leq e_2 \leq V(175)$ or $e_2 = V(728)$ is $W_1 \approx (3^6/4) \approx 182$. When R_1 is the minimum even integer that is $(W_0/W_1)^{1/2} \approx (3^6/(3^6/4))^{1/2} \approx 2 \leq 2 = R_1$, the total number that (PI) and (IM) act on $|e_3>$ is $R_1 \approx 2$. Next, (OB) observes $|e_3>$, and the data of W_1 remain. Similarly, (E_z) [$2 \leq z \leq 3$. z is the integer.] changes $|e_3>$ for $|1>$ in $V(0) \leq e_2 \leq V((3^6/4^z) - 7)$ [$z = 2$; $V((3^6/16) - 7) \approx V(39)$, $z = 3$; $V((3^6/64) - 7) \approx V(4)$] or $e_2 = V(728)$, or it changes $|e_3>$ for $|0>$ in the others of e_2. As the target state for $|e_3>$ is 1, (PI) and (IM) act on $|e_3>$. The number of the data that is included in $V(0) \leq e_2 \leq V((3^6/4^z) - 7)$ or $e_2 = V(728)$ is $W_z \approx 3^6/4^z$. When R_z is the minimum even integer that is $(W_{z-1}/W_z)^{1/2} \approx ((3^6/4^{z-1})/(3^6/4^z))^{1/2} \approx 2 \leq 2 = R_z$, the total number that (PI) and (IM) act on $|e_3>$ is $R_z \approx 2$. Next, (OB) observes $|e_3>$, and the data of W_z remain. These actions are repeated sequentially from 2 to 3 at z. (E_4) changes $|e_3>$ for $|1>$ at $e_2 = V(728)$, or it changes $|e_3>$ for $|0>$ in the others of e_2. As the target state for $|e_3>$ is 1, (PI) and (IM) act on $|e_3>$. The number of the data that is included at $e_2 = V(728)$ is $W_4 \approx 3! \approx 3^6/4^4$. When R_4 is the minimum even integer that is $(W_3/W_4)^{1/2} \approx ((3^6/4^3)/(3^6/4^4))^{1/2} \approx 2 \leq 2 = R_4$, the total number that (PI) and (IM) act on $|e_3>$ is $R_4 \approx 2$. Next, (OB) observes $|a_f>$, $|b_1>$, $|b_2>$, $|c_i>$, $|d>$, $|e_1>$, $|e_2>$ and $|e_3>$, and one of the data of W_4 remains. For example, a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, c_1, c_2, c_3, d, e_1, e_2 and e_3 are 0, 1, 2, 0, 1, 2, 1, (0, 1, 2, 0, 1, 2), 1, 1, 1, 0, 4, (2, 2, 2, 2, 2, 2) and 1, respectively, it is obtained that 3 subsets are (1, 4), (2, 5) and (3, 6).

5. Discussion and Summary

The computational complexity of this quantum algorithm [= S] becomes the following. In the order of the actions by the gates, the number of them is Pn at $\boxed{\mathrm{H}}$, n at (A), $Qn \approx 2n$ at (PI) and (IM), n at (OB), $n(n - 1)$ at (B), $K + 1$ at (C), 2 at (D), g at (E_z) [$1 \leq z \leq g$. z is the integer.], $\sum_{z=1 \to g} R_z \approx 2g$ at (PI) and (IM), and g at (OB). Therefore, S becomes $n^2 + (P + 3)n + K + 3 + 4g$. In the example of the section 4, S is 88. The computational complexity of the classical computation [= Z] is $W_0 = K^n = 3^6 = 729$. After all, S/Z becomes about 1/8. When n is large enough, S becomes about n^2, where P is about $log_2 K$, g is about $(1/2)log_2 (K^n/K!)$, and $K!$ is about $K^K e^{-K}(2\pi K)^{1/2}$ [Stirling's formula]. And then, S/Z is about n^2/K^n. For example, as for $n = 100$ and $K = 3$, S/Z is about $100^2/3^{100} \approx 1/10^{44}$.

Therefore, the polynomial time process becomes possible.

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