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On θ-Convergence and $S_{2\frac{1}{2}}$ Spaces

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Abstract

In this paper we obtain some characterizations of $S_{2\frac{1}{2}}$ spaces. We obtain conditions for a compact set and a point not in the set for having disjoint closed neighborhoods in $S_{2\frac{1}{2}}$ spaces. We have also obtained a characterization for an S_1 space to be $S_{2\frac{1}{2}}$.

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1. Introduction and Preliminaries

In [2], Dorsett introduced the concept of weakly Urysohn spaces and showed that these spaces lie strictly between the S_3 and S_2 of Császár [1] and were weaker than the well known separation axiom of Urysohn ($T_{2\frac{1}{2}}$). Also it was proved that for rim-compact spaces the concepts of S_3, S_2 and weakly Urysohn coincide. In [9], weakly Urysohn was introduced in a refined form and was utilized to obtain some map gluing theorems on sθ-continuity of maps. In [3], Dorsett obtained some characterizations of R_0 and R_1 spaces in terms of nets and closures. In [5], Dube has given several conditions for a compact set and a point outside the set to have disjoint neighborhoods in an R_1 space and proved that the set of points whose images under two given continuous functions into an R_1 space have the same closure, is closed. In [6], Dunham introduced the concept of weakly Hausdorff spaces and proved it to be equivalent to R_1 space.

As weakly Urysohn spaces are obtained from $T_{2\frac{1}{2}}$, and lie strictly between the S_3 and S_2 separation axioms we preferred to call them $S_{2\frac{1}{2}}$ spaces in [10]. In this paper analogous to weakly Hausdorff spaces we have defined weakly θ-Urysohn spaces and have proved it to be equivalent to $S_{2\frac{1}{2}}$ spaces [Theorem 2.3 below]. We have also given some conditions for a compact set and a singleton disjoint from it to have disjoint closed neighborhoods in an $S_{2\frac{1}{2}}$ space [Theorem 2.4 below] and have proved that the set of points whose images under two given continuous functions into an $S_{2\frac{1}{2}}$ space have the same closure, is θ-closed [Theorem 2.5 below]. Further a condition was obtained where the S_1 and $S_{2\frac{1}{2}}$ separation axioms coincide [Theorem 2.9 below]. A characterization of $S_{2\frac{1}{2}}$ space was obtained in terms of regular open sets [Theorem 2.10 below].

Throughout, by a space X we shall mean a topological space and $cl(A)$, $int(A)$ and A^C will denote the closure, interior the complement of A in X respectively.

For a topological space X and a subset A of X, a point $x \in X$ is said to be in the θ-closure of A denoted by $cl_\theta(A)$ if for every neighborhood U of x we have $cl(U) \cap A \neq \phi$. The subset A is θ-closed if $A = cl_\theta(A)$ and regular open if $A = int(cl(A))$ or equivalently if it is interior of some closed set. Also, a point $x \in ker(A)$ if $cl\{x\} \cap A \neq \phi$ and $< x >= cl\{x\} \cap ker\{x\}$. A point $x \in X$ is a θ-convergent point of a net P in X, if the net is eventually in the closure of every neighborhood containing x. A set which can be expressed as the closure of a singleton is said to be a point closure set. Further a space X is said to be,

1. Urysohn ($T_{2\frac{1}{2}}$) space [9] if for every pair of distinct points x and y in X there exist neighborhoods U of x and V of y such that $cl(U) \cap cl(V) = \phi$.

2. S_3 (regular) space [1] if for any point x and a closed set F not containing x there exist disjoint neighborhoods containing them.

3. $S_{2\frac{1}{2}}$ space [1] (S-S_2 in sense of [9], weakly Urysohn in sense of [2]) if for every pair of distinct points x and y in X, whenever $cl\{x\} \neq cl\{y\}$ then there exist neighborhoods U of x and V of y such that $cl(U) \cap cl(V) = \phi$.

4. S_2 space [1] (R_1 in sense of [2] [3] [5] [7] [8]) if for every pair of distinct x and y in X, whenever $cl\{x\} \neq cl\{y\}$ then there exist disjoint neighborhoods containing them.

5. S_1 space [1] (R_0 in sense of [3] [4]) if for every pair of distinct points x and y, whenever x has a neighborhood not containing y, then y has a neighborhood not containing x.

In this paper we shall use the following results.

Lemma 1.1. [8] For each $x, y \in X$, $< x >=< y >$ if and only if $cl\{x\} = cl\{y\}$ if and only if $ker\{x\} = ker\{y\}$.

Lemma 1.2. [1] In an S_2 space X and $K \subset X$ if K is compact then $cl(K)$ is compact.

Lemma 1.3. [4] For any space X the following statements are equivalent:

(a) X is S_1 .

(b) For any $x, y \in X$, $y \in cl\{x\}$ if and only if $cl\{x\} = cl\{y\}$.

Lemma 1.4. [8] For any topological space X the following statements are equivalent:

(a) X is S_2.

(b) For any $x \in X$, $cl\{x\} = cl_\theta\{x\}$.

Lemma 1.5. A map $f : X \to Y$ is continuous if and only if $cl(f^{-1}(B)) \subset f^{-1}(cl(B))$ for all $B \subset Y$.

2. Results

Definition 2.1. A space X is weakly θ-Urysohn if and only if for any $x, y \in X$, $cl\{x\} = cl\{y\}$ whenever there is a net $P : D \to X$ in X which θ-converges to both x and y.

Theorem 2.2. X is weakly θ-Urysohn if and only if for each $x, y \in X$ one of the following holds.

(a) $cl\{x\} = cl\{y\}$.

(b) There exist neighborhoods U of x and V of y such that $cl(U) \cap cl(V) = \phi$.

Proof. The proof is similar to [6, Theorem 2.2], and hence is omitted. ■

Theorem 2.3. X is weakly θ-Urysohn if and only if $S_{\frac{1}{2}}$.

Proof. The proof follows from Theorem 2.2 and is similar to [6, Theorem 2.7] and hence is omitted. ■

Theorem 2.4. In an $S_{2\frac{1}{2}}$ space, for a compact set F and $x \in X$ such that $x \notin F$, have disjoint closed neighborhoods for the following cases:

(a) $x \notin cl\{F\}$.

(b) $y \in F$ implies $cl\{y\} \subset F$.

(c) $y \in F$ implies $ker\{x\} \neq ker\{y\}$.

(d) $y \in F$ implies $cl\{x\} \neq cl\{y\}$.

(e) $y \in F$ implies $< x > \neq < y >$.

Proof. The proof follows from Lemma 1.1 and Lemma 1.2, is straightforward and hence is omitted. ■

Theorem 2.5. If f and g are two continuous functions from a topological space (X, T) to an $S_{2\frac{1}{2}}$ topological space $(X*, T*)$ then the set $A = \{x \in X : clf(x) = clg(x)\}$ is θ-closed in X.

Proof. The proof follows from Lemma 1.5 is similar to [5, Theorem 3.5] and hence is omitted. ■

In [7], it is proved that a space X is S_2 is equivalent to the condition that for any $x, y \in X$, $y \in cl_\theta\{x\}$ if and only if every net which converges to y converges to x. We prove that in an S_2 space for any points $x, y \in X$, y is in the θ-closure of x if and only if every net which θ-converges to y θ-converges to x. We also show that unlike above in [7] this condition in terms of θ-convergence is not equivalent for a space to be S_2.

Theorem 2.6. For any S_2 space X and $x, y \in X$, $y \in cl_\theta\{x\}$ if and only if every net which θ-converges to y θ-converges to x.

Proof. Let $y \in cl_\theta\{x\}$.Let $P : D \to X$ be a net in X such that P θ-converges to y. Since $y \in cl_\theta\{x\}$ implies $x \in cl_\theta\{y\}$. As space X is S_2, by Lemma 1.4 we have $x \in cl\{y\}$. Thus P θ-converges to x.

Conversely, let every net which θ-converges to y θ-converges to x. Let $P = y$, be a constant net in X which θ-converges to y and thus θ-converges to x which implies $x \in cl_\theta\{y\}$ and hence $y \in cl_\theta\{x\}$. ■

The following example shows that the converse of Theorem 2.6 does not hold.

Example 2.7. Let $X = \{0, 1\}$, together with the topology $T = \{\phi, \{0\}, X\}$. Such a topological space is well known as Siperinski space. This space satisfies the condition that for any $x, y \in X$, $y \in cl_\theta\{x\}$ if and only if every net which θ-converges to y θ-converges to x. But since $cl\{0\} \neq cl\{1\}$ and $\{0\}, \{1\}$ do not have disjoint neighborhoods, the space is not S_2.

In the next example we show that Theorem 2.6 does not hold if the space is not S_2.

Example 2.8. Let $X = \mathbf{Z}^+$, the set of all positive integers together with the topology $T = \{G \subset X : 0 \notin G\}$ i.e. all those sets are open which do not contain the point 0. Let $x \in X$ be any point distinct from 0. Then as $cl\{0\} \neq cl\{x\}$ and $\{0\}, \{x\}$ do not have disjoint neighborhoods, the space is not S_2. As, $\{0\} \in cl_\theta\{1\}$ and a sequence and hence a net defined as $\{1, 2, 3, 4 \ldots\}$ θ-converges to 0 but does not θ-converges to 1, Theorem 2.6 does not hold.

Theorem 2.9. Any S_1 topological space X is $S_{2\frac{1}{2}}$ if and only if for every θ-convergent net P in X the set of all θ-convergent points of the net P, is a point closure set.

Proof. Let X be any S_1 space which is also $S_{2\frac{1}{2}}$. Let P be a θ-convergent net in X such that P θ-converges to some point $x \in X$. Let G be the set of all θ-convergent points of the net P and let $y \in G$. Then by Theorem 2.3 $cl\{x\} = cl\{y\}$ which implies $y \in cl\{x\}$ and thus $G \subset cl\{x\}$. If $y \in cl\{x\} \subset cl_\theta\{x\}$ then $x \in cl_\theta\{y\}$ and since every $S_{2\frac{1}{2}}$ is S_2 by Theorem 2.6 we have P θ-converges to y. Thus, $y \in G$ and it implies $cl\{x\} \subset G$. Therefore, $G = cl\{x\}$ and is a point closure set.

Converse follows from Lemma 1.3 and Theorem 2.3. ■

Theorem 2.10. For any topological space X the following statements are equivalent:

(a) X is $S_{2\frac{1}{2}}$.

(b) If $x, y \in X$ such that $cl\{x\} \neq cl\{y\}$ then there exists regular open sets U_1 and U_2 such that $x \in U_1$, $y \notin U_1$, $y \in U_2$, $x \notin U_2$ such that $X = U_1 \cup U_2$.

Proof. $(a) \Rightarrow (b)$ Let X is an $S_{2\frac{1}{2}}$ space. Let for any $x, y \in X$ be such that $cl\{x\} \neq cl\{y\}$.Then as space X is $S_{2\frac{1}{2}}$, there exist open sets V containing x and W containing y such that, $cl(V) \cap cl(W) = \phi$. Then, $int(W^C) = U_1$ and $int(V^C) = U_2$ being interior of closed sets are regular open sets such that $x \in U_1$, $y \notin U_1$, $y \in U_2$, $x \notin U_2$ such that $X = U_1 \cup U_2$.

$(b) \Rightarrow (a)$ Let $x, y \in X$ be such that $cl\{x\} \neq cl\{y\}$. Then there exists regular open sets U_1 and U_2 such that $x \in U_1$, $y \notin U_1$, $y \in U_2$, $x \notin U_2$ such that $X = U_1 \cup U_2$. Since regular open sets are open and $(U_1)^C \cap (U_2)^C = \phi \Rightarrow (int(cl(U_1)))^C \cap (int(cl(U_2)))^C = \phi \Rightarrow cl(int(U_1{}^C)) \cap cl(int(U_2{}^C)) = \phi \Rightarrow cl(int(U_2)) \cap cl(int(U_1)) = \phi \Rightarrow cl(U_2) \cap cl(U_1) = \phi$. Therefore, there exist open sets U_1 containing x and U_2 containing y such that, $cl(U_1) \cap cl(U_2) = \phi$ and hence the space X is $S_{2\frac{1}{2}}$. ■

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