

## The Centralizer Graph of Finite Non-abelian Groups

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### Abstract

Let  $G$  be a finite non-abelian group. In this paper, we introduce a new graph called the centralizer graph, denoted as  $\Gamma_{\text{cent}}$ . The vertices of this graph are proper centralizers in which two vertices are adjacent if their cardinalities are identical. The centralizer graph of dihedral groups, quaternion groups and dicyclic groups is found.

### 1. Introduction

Graph theory is the study of points (vertices) and lines (edges). More precisely, it involves the ways in which sets of points can be connected by lines or arcs. The concept in graph theory is widely used among many fields and one of these uses is in group theory. Algebraic graph theory is a bridge between group and graph theory and many researches have been done on this topic. In this section, we state some basic definitions that are needed in this paper, starting with some definitions related to graph theory that can be found in one of the references ([1] and [2]).

**Definition 1.1 [1]** A graph  $\Gamma$  is a mathematical structure consisting of two sets namely vertices and edges which are denoted by  $V(\Gamma)$  and  $E(\Gamma)$ , respectively. The graph is called directed if its edges are identified with ordered pair of vertices. Otherwise,  $\Gamma$  is called undirected. Two vertices are adjacent if they are linked by an edge.

**Definition 1.2 [1]** A connected graph is a graph in which there is a partition of vertex  $V$  into non empty subsets,  $V_1, V_2, \dots, V_n$  such that two vertices  $\omega_1$  and  $\omega_2$  are connected if and only if they belong to the same set  $V_i$ . Subgraphs  $\Gamma(V_1), \Gamma(V_2), \dots, \Gamma(V_n)$  are all components of  $\Gamma$ . The graph  $\Gamma$  is connected if it has precisely one component.

**Definition 1.3 [1]** A subgraph of a graph  $\Gamma$  is a graph whose vertices and edges are subset of the vertices and edges of  $\Gamma$ . Hence we denote  $\Gamma_{\text{sub}}$  a subgraph of  $\Gamma$ .

**Definition 1.4 [2]** A complete graph is a graph where each order pair of distinct vertices are adjacent, and it is denoted by  $K_n$ .

Next, we state some graph properties that are needed in this paper: A nonempty set  $S$  of  $V(\Gamma)$  is called an independent set of  $\Gamma$  if there is no adjacent between two elements of  $S$  in  $\Gamma$ , whilst the independent number is the number of vertices in maximum independent set and it is denoted by  $\alpha(\Gamma)$ . However, the maximum number  $c$  for which  $\Gamma$  is  $c$ -vertex colorable is known as chromatic number and is denoted by  $\chi(\Gamma)$ . The diameter is the maximum distance between any two vertices of  $\Gamma$  and  $d(\Gamma)$  is used as a notation. Furthermore, a clique is a complete subgraph in  $\Gamma$ , while the clique number is the size of the largest clique in  $\Gamma$  and is denoted by  $\omega(\Gamma)$ . The dominating set  $X \subseteq V(\Gamma)$  is a set where for each  $v$  outside  $X$ , there exists  $x$  in  $X$  such that  $v$  adjacent to  $x$ . The minimum size of  $X$  is called the dominating number denoted by  $\gamma(\Gamma)$  ([1] and [2]).

Since this work is a connection between group theory and graph theory, hence we provide some works related to group theory, more precisely to the commutativity degree in terms of centralizers.

The commutativity degree is the probability that two random elements of a group commute [3]. This concept has then been generalized by several authors. One of the generalizations is the commutativity degree in terms of centralizers introduced by Omer *et al.* [4], where the centralizers can be used to compute the commutativity degree. In this paper, we find a connection between the group theory and graph theory by defining a new graph related to centralizers, namely the centralizer graph.

This paper is structured as follows: In Section 2, we state some previous works that are related to the graph. Whilst, our main results are introduced in Section 3 which include results on the centralizer graph and some results on centralizer graph of dihedral groups, quaternion groups and its generalized group, namely dicyclic groups.

## 2. Preliminaries

In this section, some earlier and recent works needed in this paper are included, starting with some kinds of graphs, followed by the commutativity degree in terms of centralizers.

In 1975, a new graph was introduced, namely a non-commuting graph [6]. The definition of a non-commuting graph is stated in the following.

**Definition 2.1 [6]** Let  $G$  be a finite non-abelian group with the center denoted by  $Z(G)$ . A non-commuting graph is a graph whose vertices are non central elements of  $G$  (i.e  $G - Z(G)$ ). Two vertices  $v_1$  and  $v_2$  are adjacent whenever  $v_1 v_2 \neq v_2 v_1$ .

In [6], it is mentioned that this concept was firstly introduced by Paul Erdős. Erdős posed a question if there is a finite bound on the cardinalities of cliques of  $\Gamma$ . The first confirm of Erdős's question was by Neumann [6]. According to Neumann [6] there is a finite complete subgraph in some groups. Furthermore, Abdollahi *et al.* [7] emphasized the existence of finite bound on the cardinalities of complete subgraph in  $\Gamma$ . They used the graph theoretical concepts to investigate the algebraic properties of the graph. Recently, the non-commuting graph was generalized by Erfanian and Tölue [8] to what is called the relative non-commuting graph. The idea of this generalized graph comes from the relative commutativity degree where the vertices of this graph are the group minus centralizer of its subgroup. Two vertices are linked by an edge if their commutator is not equal to one.

Moghaddamfar *et al.* [9] conjectured that for some finite non-abelian groups  $G_1$  and  $G_2$ , if the non-commuting graph of  $G_1$  is isomorphic to the non-commuting graph of  $G_2$ , then  $|G_1|=|G_2|$ . Besides, they obtained some graph properties for this conjecture. However, Moghaddamfar [10] gave some counterexamples for the conjecture mentioned in [9].

In 1990, Bertram [11] introduced a graph which is called a graph related to conjugacy classes. The vertices of this graph are non-central conjugacy classes, where two vertices are adjacent if the cardinalities are not coprime.

Next, we provide some information about the commutativity degree in terms of centralizers.

The commutativity degree is the probability that two random elements of a group commute [3]. This concept has then been generalized by several authors. One of the generalizations is the commutativity degree in terms of centralizers that was firstly investigated by Omer *et al.* [4] who worked on dihedral groups. One of their results is given in the following.

**Proposition 2.2 [4]** Let  $G$  be a dihedral group of order  $2n$ . If  $G$  acts on itself by conjugation, then the commutativity degree in terms of centralizers is equal to  $\frac{\text{Cent}(G)+1}{|G|}$ .

### 3. Main Results

In this section, we introduce our main results, start by a new definition.

**Definition 3.1** Let  $G$  be a finite non-Abelian group. The centralizers graph, denoted by  $\Gamma_{\text{cent}}$ , is a graph whose vertices are proper centralizers,  $|V(\Gamma_{\text{cent}})|=C(G)-A$ , where  $C(G)$  is the number of proper centralizers in  $G$  and  $A$  is the number of improper centralizers in  $G$ . Two vertices of  $\Gamma_{\text{cent}}$  are joined by an edge if their cardinalities are identical.

Next, some basic results are provided.

**Proposition 3.2** Let  $G$  be a finite group, and  $\Gamma_{\text{cent}}$  its centralizer graph. If  $G$  is an Abelian group, then  $\Gamma_{\text{cent}}$  is null.

**Proof 3.3** If  $G$  is an Abelian group, then  $Z(G) = G$ . Thus  $C_G(x) = x$  for all  $x \in G$ . Using Definition 3.1, the number of vertices  $|V(\Gamma_{\text{cent}})| = C(G) - A = 0$ . thus the graph  $\Gamma_{\text{cent}}$  is null. ■

**Proposition 3.4** Let  $\Gamma_{\text{cent}}$  be the centralizer graph of finite non-Abelian group  $G$ . Then  $\Gamma_{\text{cent}}$  is not connected.

**Proof 3.5** Suppose that  $\Gamma_{\text{cent}}$  is a connected graph. Assume that  $x, y \in V(\Gamma_{\text{cent}})$  such that  $|x| = |y|$  and that  $x$  is adjacent to  $y$ . Therefore there exists  $k \in G$  such that  $[G : k] = 2$ . Hence,  $k$  is not adjacent to  $x$  or  $y$ , a contradiction. Therefore,  $\Gamma_{\text{cent}}$  is not connected. ■

In the following section, the graph related to centralizers of some groups is found.

### 3.1 Centralizer Graph of Dihedral Groups, Quaternion Groups and Dicyclic Groups

In this section, we find the centralizer graph of dihedral group. Our results are stated in the following.

**Theorem 3.6** Let  $G$  be a dihedral group of order  $2n$ . If  $G$  acts on itself by conjugation, then

$$\Gamma_{\text{cent}} = \begin{cases} K_{\frac{n}{2}}, & n \text{ is even,} \\ K_n, & n \text{ is odd.} \end{cases}$$

**Proof 3.7** If  $G$  acts on itself by conjugation, then there exists  $\varphi : G \times G \rightarrow G$  such that  $\varphi_g(a) = gag^{-1}$  where  $a, g \in G$ . Thus the centralizer of  $a$  in  $G$  is  $C_g(a) = \{gag^{-1} = a, g \in G\}$ . In accordance with Theorem 3.1 in [4], the number of centralizers in  $G$  are  $\frac{n}{2} + 2$  if  $n$  is even and  $n + 2$  if  $n$  is odd. Since the vertices of  $\Gamma_{\text{cent}}$  are non proper centralizers, thus the number of vertices in  $\Gamma_{\text{cent}}$  is

$$|V(\Gamma_{\text{cent}})| = \begin{cases} \frac{n}{2} + 1, & n \text{ is even,} \\ n + 1, & n \text{ is odd.} \end{cases}$$

According to Theorem 3.1 in [4], the number of centralizers in  $G$  that have the same sizes are  $\frac{n}{2}$ , where  $n$  is even and equal to  $n$  if  $n$  is odd. Therefore, there are one complete component of  $K_{\frac{n}{2}}$ , when  $n$  is even, and  $K_n$  when  $n$  is odd. The proof then follows. ■

Next, the centralizer graph of a quaternion group is given.

**Theorem 3.8** Let  $G$  be a quaternion group. If  $G$  acts on itself by conjugation, then  $\Gamma_{\text{cent}} = K_{2^{n-2}}$ .

**Proof 3.9** Since the number of conjugacy classes in  $G$  is  $2^{n-2} + 3$ , the number of centralizers in  $G$ ,  $C(G) = 2^{n-2} + 2$ . By Definition 3.1,  $|V(\Gamma_{\text{cent}})| = 2^{n-2} + 1$ . According to adjacency of vertices, there are  $2^{n-2}$  centralizers of size four. It follows that  $\Gamma_{\text{cent}}$  consists of a complete component of  $K_{2^{n-2}}$  and one isolated vertex. The proof then follows. ■

Finally, the centralizers graph is found for dicyclic groups.

**Theorem 3.10** Let  $G$  be a finite dicyclic group,  $G \cong \langle a, b : a^{2^{\beta}}, b^4 = 1, b^{-1}ab = a^{-1}, a^{\beta} = b^2 \rangle$ . If  $G$  acts on itself by conjugation, then  $\Gamma_{\text{cent}} = K_{2^{n-2}}$ .

**Proof 3.11** The proof is similar to the proof of Theorem 3.8.

#### 4. Conclusion

In this paper, we introduced a new graph, namely the centralizer graph whose vertices are proper centralizers of a group. Besides, the centralizer graph is found for dihedral groups and quaternion groups.

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