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Fixed point theorems on g-reciprocal continuity via absorbing maps

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Abstract

The objective of this paper is to present common fixed point theorems under extended strict contractive condition by using g-reciprocal continuity and absorbing maps. These results extend and generalize the recent results of R.U. Joshi et al. [5] and we illustrate these results by suitable examples.

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1. Introduction and Preliminaries

The study of common fixed points under strict contractive conditions using noncompatible mappings was initiated by Pant [2,3]. But the study of strict contractive conditions do not ensure the existence of common fixed points, unless very strong conditions like compactness are assumed. However, the concept of g-reciprocal continuity, a new analogue of reciprocal continuity ensures the existence of common fixed points without assuming any strong conditions on the space or on the mappings. Many works under the analogue

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of reciprocal continuity have come through from past few years(see [8]-[11] and the references therein). Recently, Pant et al. [4], have introduced this notion of g-reciprocal continuity, which ensures the existence of common fixed point under strict contractive conditions without assuming any stronger conditions on the space. Following this work, R.U. Joshi et al. [5] have proved a fixed point theorem for a generalized strict contractive condition, wherein the mappings involved are assumed to be g-compatible which enforce commutativity at its coincidence points.

In the present paper, we use the notion of absorbing maps [6], which are neither a subclass of compatible maps nor a subclass of noncompatible maps, as they do not enforce commutativity at its coincidence points. Further these results are the extension and generalization of the result of R.U. Joshi et al. and many more results in the literature.

Definition 1.1. [1] Two self maps f and g of a metric space *(X,d)* are called compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some t in X. Thus the mappings f and g will be noncompatible if there exists at least one sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some t in X but

$$\lim_{n\to\infty} d(fgx_n, gfx_n)$$

is either nonzero or nonexistent.

Definition 1.2. [3] Two self mappings f and g of a metric space (X,d) are called reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some t in X.

Definition 1.3. [4] Two self mappings f and g of a metric space (X,d) are called g-reciprocally continuous iff $\lim_{n\to\infty} ffx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some t in X.

Definition 1.4. Two self mappings f and g of a metric space (X, d) are called g-compatible if $\lim_{n\to\infty} d(ffx_n, gf_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some t in X.

Definition 1.5. [6] A pair of self mappings *(f,g)* of a metric space (X, d) is called g-absorbing if there exists some real number $R > 0$ such that $d(gx, gfx) \leq Rd(fx, gx)$ for all x in X.

Example 1.6. Let $X = [0, 1]$, d be the usual matric on X. Define $f, g : X \to X$ by $fx = 1$ if $x \neq 1$, $f1 = 0$ and $gx = 1$ for all x. Then one can easily verify that f is g-absorbing but it is not g-compatible.

In 2002, M. Aamri and D. El Moutawakil [7] introduced the property (E.A.), which is a true generalization of non compatible maps in metric spaces. For works on (E.A.) property refer [12-13].

Definition 1.7. [7] Let f, g be two self mappings of a metric space (X, d). Then we say that f and g satisfy the property (E.A.), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

Definition 1.8. Let (f, g) and (S, T) be two pairs of self mappings of a metric space (X, d). Then we say that (f, g) and (S, T) satisfies the common (E.A.) property, if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = t$$

for some $t \in X$.

2. Main Result

We now state and prove our first main result.

Theorem 2.1. Let f, g and S be three self mappings of a metric space (X,d) satisfying

1. Common (E.A.) Property
2. $d(fx, Sy) < max\left\{d(gx, gy), \frac{d(fx, gx) + d(Sy, gy)}{2}, \frac{d(fx, gy) + d(Sy, gx)}{2}\right\}$ $\forall x \neq y$.

3. f and S are g-absorbing.

If one of the pair (f,g) or (S,g) is g-reciprocally continuous, then f, g and S have a unique common fixed point.

Proof. Given that (f, g) and (S, g) satisfies Common (E.A.) Property. Then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} gy_n = t$$

for some $t \in X$.

If (f, g) is g-reciprocally continuous, then $ffx_n \to ft$ and $gfx_n \to gt$ as $n \to \infty$. Since f is g-absorbing, there exists $R_1 > 0$ such that

$$d(gx, gfx) \leq R_1\, d(fx, gx) \quad \text{for all } x \in X.$$

Consider $d(gx_n, gfx_n) \leq R_1 d(fx_n, gx_n)$. On letting $n \to \infty$ we obtain $gt = t$. Now we will prove that $ft = t$. Consider

$$d(ffx_n, Sy_n) < max\left\{d(gfx_n, gy_n), \frac{d(ffx_n, gfx_n) + d(Sy_n, gy_n)}{2}, \frac{d(ffx_n, gy_n) + d(Sy_n, gfx_n)}{2}\right\}$$

letting $n \to \infty$ we obtain $d(ft, t) \leq \dfrac{d(ft, t)}{2}$ which gives $ft = t = gt$.

Similarly consider

$$d(ffx_n, St) < max\left\{d(gfx_n, gt), \frac{d(ffx_n, gfx_n) + d(St, gt)}{2}, \frac{d(ffx_n, gt) + d(St, gfx_n)}{2}\right\}$$

letting $n \to \infty$ we obtain $ft = St = gt = t$. Thus t is a common fixed point of f, g and S.

Next, suppose that (S, g) is g-reciprocally continuous, then $SSy_n \to St$ and $gSy_n \to gt$ as $n \to \infty$. Since S is g-absorbing, there exists $R_2 > 0$ such that

$$d(gx, gSx) \leq R_2 d(Sx, gx) \quad \text{for all } x \in X.$$

Consider $d(gy_n, gSy_n) \leq R_2 d(Sy_n, gy_n)$. On letting $n \to \infty$ we obtain $gt = t$. Now we will prove that $St = t$. Consider

$$d(fx_n, SSy_n) < max\left\{d(gx_n, gSy_n), \frac{d(fx_n, gx_n) + d(SSy_n, gSy_n)}{2}, \frac{d(fx_n, gSy_n) + d(SSy_n, gx_n)}{2}\right\}$$

letting $n \to \infty$ we obtain $d(t, St) \leq \dfrac{d(St, t)}{2}$ which gives $St = t = gt$. Similarly consider

$$d(ft, SSy_n) < max\left\{d(gt, gSy_n), \frac{d(ft, gt) + d(SSy_n, gSy_n)}{2}, \frac{d(ft, gSy_n) + d(SSy_n, gt)}{2}\right\}$$

letting $n \to \infty$ we obtain $ft = St = gt = t$. Thus t is a common fixed point of f, g and S.

Uniqueness: Let u and v be the two common fixed points of f, g and S. Then $u = fu = gu = Su$ and $v = fv = gv = Sv$. Now we have to prove that $u = v$. Suppose that $u \neq v$. Then,

$$d(u, v) = d(fu, Sv) < max\left\{d(gu, gv), \frac{d(fu, gu) + d(Sv, gv)}{2}, \frac{d(fu, gv) + d(Sv, gu)}{2}\right\}$$

which gives $d(u, v) < d(u, v)$, a contradiction. Hence $u = v$. ■

The above theorem is illustrated by the following example.

Example 2.2. Let $X = [2, 10]$ and d be the usual metric on X. Define $f, g, S : X \to X$ by

$$Sx = 2 \quad \forall x \in X \quad fx = \begin{cases} 2 & \text{if } x = 2 \text{ or } x > 5 \\ 4 & \text{if } 2 < x \leq 5 \end{cases} \qquad gx = \begin{cases} 2 & \text{if } x = 2 \\ 6 & \text{if } 2 < x \leq 5 \\ \dfrac{x+1}{3} & \text{if } x > 5 \end{cases}$$

The mappings (f, g) and (S, g) are g-reciprocally continuous. To see this, let $\{x_n\} = \left\{5 + \dfrac{1}{n}\right\}$ and $\{y_n\} = \left\{5 + \dfrac{2}{n}\right\}$ be two sequences in X where $\dfrac{1}{n} \to 0$ as $n \to \infty$. Then $fx_n = 2$, $gx_n = 2 + \dfrac{1}{3n} \to 2$, $Sy_n = 2$, $gy_n = 2 + \dfrac{2}{3n} \to 2$ as $n \to \infty$. $ffx_n = f(2) = 2$, $gfx_n = g(2) = 2$, $SSy_n = S(2) = 2$ and $gSy_n = g(2) = 2$. Thus (f, g) and (S, g) satisfies the common (E.A.) property and are g-reciprocally continuous. Also for all $x \in X$ we have

$$d(gx, gfx) \leq R_1 d(fx, gx) \quad \text{and} \quad d(gx, gSx) \leq R_2 d(Sx, gx) \quad \text{where } R_1, R_2 \geq 1.$$

Therefore f and S are g-absorbing. Further, f, g and S satisfy all the conditions of Theorem 2.1 and have a unique common fixed point at $x = 2$.

As a corollary of Theorem 2.1, we derive the following fixed point result for two self mappings, which is a sharpened version of Theorem 3.1 contained in R.U. Joshi [5].

Corollary 2.3. Let f and g be g-reciprocally continuous self mappings of a metric space (X,d) satisfying

1. (E.A.) Property
2. $d(fx, fy) < max\left\{d(gx, gy), \dfrac{d(fx, gx) + d(fy, gy)}{2}, \dfrac{d(fx, gy) + d(fy, gx)}{2}\right\}$ $\forall x \neq y$.
3. f is g-absorbing.

Then f and g have a unique common fixed point.

Proof. Put $S = f$ in Theorem 2.1. ∎

Example 2.4. Let $X = [1, 10]$ and d be the usual metric on X. Define $f, g : X \to X$ by

$$fx = \begin{cases} 5 & \text{if } x \leq 5 \\ 4 & \text{if } x > 5 \end{cases} \qquad gx = \begin{cases} 10 & \text{if } x > 5 \\ \dfrac{x+5}{2} & \text{if } x \leq 5 \end{cases}$$

The mappings (f, g) is g-reciprocally continuous. To see this, let $\{x_n\} = \left\{5 - \dfrac{1}{n}\right\}$ be a sequence in X where $\dfrac{1}{n} \to 0$ as $n \to \infty$. Then

$$fx_n = 5, gx_n = 5 - \frac{1}{2n} \to 5 \text{ as } n \to \infty, ffx_n = f(5) = 5, gfx_n = g(5) = 5.$$

Thus the pair (f, g) satisfies the (E.A.) property and is g-reciprocally continuous. Also

$$d(gx, gfx) \leq Rd(fx, gx) \quad \text{for all } x \in X \text{ where } R \geq 1.$$

Therefore f is g-absorbing. Further, f and g satisfies all the conditions of Corollary 2.2 and have a unique common fixed point at $x = 5$.

Theorem 2.5. Let f, g, S and T be four self mappings of a metric space (X,d) satisfying

1. Common (E.A.) Property
2. $d(fx, Sy) < max\left\{d(gx, Ty), \dfrac{d(fx, gx) + d(Sy, Ty)}{2}, \dfrac{d(fx, Ty) + d(Sy, gx)}{2}\right\}$ $\forall x \neq y$.
3. f is g-absorbing and S is T-absorbing.

4. (f,g) and (S,T) are g-reciprocally continuous

Then f, g, S and T have a unique common fixed point.

Proof. Given that (f, g) and (S, T) satisfies Common (E.A.) Property. Then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = t \quad \text{for some } t \in X.$$

Since f is g-absorbing and S is T-absorbing, there exists $R_1, R_2 > 0$ such that

$$d(gx, gfx) \le R_1 d(fx, gx) \quad \text{and} \quad d(Tx, TSx) \le R_2 d(Sx, Tx) \quad \text{for all } x \in X.$$

(f, g) and (S, T) are g-reciprocally continuous implies

$$ffx_n \to ft, gfx_n \to gt \quad \text{and} \quad SSy_n \to St \quad \text{and}$$
$$TSy_n \to Tt \quad \text{as} \quad n \to \infty.$$

Consider $d(gx_n, gfx_n) \le R_1 d(fx_n, gx_n)$. On letting $n \to \infty$ we obtain $gt = t$.

Similarly $d(Ty_n, TSy_n) \le R_2 d(Sy_n, Ty_n)$ gives $Tt = t$. Now we will prove that $ft = t$. Consider

$$d(ffx_n, Sy_n) < max\{d(gfx_n, Ty_n), \frac{d(ffx_n, gfx_n) + d(Sy_n, Ty_n)}{2}, \frac{d(ffx_n, Ty_n) + d(Sy_n, gfx_n)}{2}\}$$

Letting $n \to \infty$ we obtain $d(ft, t) \le \frac{d(ft, t)}{2}$ which gives $ft = t = gt$.

Again Consider

$$d(fx_n, SSy_n) < max\{d(gx_n, TSy_n), \frac{d(fx_n, gx_n) + d(SSy_n, TSy_n)}{2}, \frac{d(fx_n, TSy_n) + d(SSy_n, gx_n)}{2}\}$$

letting $n \to \infty$ we obtain $d(t, St) \le \frac{d(St, t)}{2}$ which gives $St = t = Tt$. Hence $ft = gt = St = Tt = t$. i.e. t is a common fixed point of f, g, S and T.

Uniqueness: Let u and v be the two common fixed points of f, g, S and T. Then,

$$u = fu = gu = Su = Tu \quad \text{and} \quad v = fv = gv = Sv = Tv$$

Now we have to prove that $u = v$. Suppose that $u \neq v$. Then,

$$\begin{aligned} d(u, v) &= d(fu, Sv) \\ &< max\left\{d(gu, Tv), \frac{d(fu, gu) + d(Sv, Tv)}{2}, \frac{d(fu, Tv) + d(Sv, gu)}{2}\right\} \end{aligned}$$

which gives $d(u, v) < d(u, v)$, a contradiction. Hence $u = v$. ■

Now we present an example to illustrate Theorem 2.3.

Example 2.6. Let $X = [1, 10]$ and d be the usual metric on X. Define $f, g, S, T : X \to X$ by

$$fx = \begin{cases} 3 & \text{if } x \leq 3 \\ 4 & \text{if } x > 3 \end{cases} \qquad gx = \begin{cases} 6 - x & \text{if } x \leq 3 \\ 1 & \text{if } x > 3 \end{cases}$$

$$Sx = \begin{cases} 2 & \text{if } x < 3 \\ \dfrac{x+3}{2} & \text{if } x \geq 3 \end{cases} \qquad Tx = \begin{cases} 9 & \text{if } x < 3 \\ \dfrac{2x+3}{3} & \text{if } x \geq 3 \end{cases}$$

The mappings (f, g) and (S, T) are g-reciprocally continuous. To see this, let $\{x_n\} = \left\{3 - \frac{1}{n}\right\}$ and $\{y_n\} = \left\{3 + \frac{1}{n}\right\}$ be two sequences in X where $\frac{1}{n} \to 0$ as $n \to \infty$. Then as $n \to \infty$,

$$fx_n = 3,\ gx_n = 3 + \frac{1}{n} \to 3,\ Sy_n = 3 + \frac{1}{2n} \to 3\ ,\ Ty_n = 3 + \frac{2}{3n} \to 3$$
$$ffx_n = f(3) = 3,\ gfx_n = g(3) = 3,\ SSy_n = 3 + \frac{1}{4n} \to 3 \text{ and } TSy_n = 3 + \frac{1}{3n} \to 3$$

Therefore (f, g) and (S, T) satisfies common (E.A.) property and are g-reciprocally continuous. Also $d(gx, gfx) \leq R_1 d(fx, gx)$ and $d(Tx, TSx) \leq R_2 d(Sx, Tx)$ for all $x \in X$ where $R_1 \geq 1$, $R_2 \geq 2$. Therefore f is g-absorbing and S is T-absorbing.

Further, f, g, S and T satisfies all the conditions of Theorem 2.3 and have a unique common fixed point at $x = 3$.

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