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More on Semi-Inner Products

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Abstract

Recently, the (i-s) semi-inner product was introduced by Liftaj and Teliti. We see that the (i-s) semi-inner product is the arithmetic mean between the superior semi-inner product and the interior semi-inner product. In this paper, we give some properties on the (i-s) semi-inner product.

Keywords: semi-inner product, (i-s) semi-inner product, superior semi-inner product, inferior semi-inner

1. Introduction

In 1961, Lumer [4] introduced a generalization of inner product, called “semi-inner product”, as follows.

A function [,]: $X \times X \to R$ is said to be a semi-inner product if it satisfies the following properties.

For any $x \in X$, if $[x, x] \geq 0 = [x, x]$ then $x = 0$.

$[\alpha x + \beta y, z] = \lambda[x, z] + \beta[y, x]$, for all $\alpha, \beta \in R$ and x, y, z $\in X$.

$[x, \alpha y] = \alpha[x, y]$, for all $\alpha \in R$ and $x, y \in X$.

) $|[x, y]|^2 \leq [x, x] \cdot [y, y]$, for all $x, y \in X$.

In 2004, Dragomir [1] introduced the superior semi-inner product $(x, y)_s$ and the inferior semi-inner product $(x, y)_i$ as follows.

$$(x, y)_s = \lim_{t \to 0^+} \frac{\|y + tx\|^2 - \|x\|^2}{2t}.$$

$$(x, y)_i = \lim_{t \to 0^-} \frac{\|y + tx\|^2 - \|x\|^2}{2t}.$$

This implies that, for any $x, y \in X$,

$(x, y)_i \le [x, y] \le (x, y)_s$.

Recently, Liftaj and Teliti [3] introduced the (i-s) semi-inner product $(x, y)_{i\text{-}s}$ as follows.

$$(x, y)_{i\text{-}s} = \lim_{t \to 0} \frac{\|y + tx\|^2 - \|y - tx\|^2}{4t}.$$

This implies that, for any $x, y \in X$,

$$(x, y)_{i\text{-}s} = \frac{1}{2}[\,(x, y)_i + (x, y)_s\,].$$

In this paper, we give some properties on the (i-s) semi-inner product.

2. Preliminaries

Proposition 2.1. [3] Let $x \in X$. Then
$(x, x)_{i\text{-}s} = \| x \|^2$.

Proposition 2.2. [3] Let $x, y \in X$ and let $\alpha, \beta \in R$. Then
$(\alpha x, \beta y)_{i\text{-}s} = \alpha\beta(x, y)_{i\text{-}s}$.

Proposition 2.3. [3] Let $x, y \in X$. Then

$$(x, y)_{i\text{-}s} = \frac{\inf\{[x, y]\} + \sup\{[x, y]\}}{2}.$$

Proposition 2.4. [3] Let $(X,\| \|)$ be a normed space, where $\| \|$ is a norm satisfying the parallelogram rule. Then semi-inner product $(x, y)_{i\text{-}s}$ is an inner product.

3. Main Results

Theorem 3.1. Let $x_1, x_2, \ldots x_n, y \in X$. Then
$(x_1 + x_2 + \ldots + x_n, y)_{i\text{-}s} \le \| y \| [\, \| x_1 \| + \| x_2 \| + \ldots + \| x_{n-1} \|\,] + (x_n, y)_{i\text{-}s}$.

Proof. In [5], we can conclude that
$(x_1 + x_2 + \ldots + x_n, y)_i \le \| y \| [\, \| x_1 \| + \| x_2 \| + \ldots + \| x_{n-1} \|\,] + (x_n, y)_i$
and
$(x_1 + x_2 + \ldots + x_n, y)_s \le \| y \| [\, \| x_1 \| + \| x_2 \| + \ldots + \| x_{n-1} \|\,] + (x_n, y)_s$.

Thus, $2(x_1 + x_2 + \ldots + x_n, y)_{i\text{-}s}$
$= (x_1 + x_2 + \ldots + x_n, y)_i + (x_1 + x_2 + \ldots + x_n, y)_s$
$\le \| y \| [\, \| x_1 \| + \| x_2 \| + \ldots + \| x_{n-1} \|\,] + (x_n, y)_i$
$+ \| y \| [\, \| x_1 \| + \| x_2 \| + \ldots + \| x_{n-1} \|\,] + (x_n, y)_s$
$= 2\| y \| [\, \| x_1 \| + \| x_2 \| + \ldots + \| x_{n-1} \|\,] + [\,(x_n, y)_i + (x_n, y)_s\,]$
$= 2\| y \| [\, \| x_1 \| + \| x_2 \| + \ldots + \| x_{n-1} \|\,] + 2(x_n, y)_{i\text{-}s}$.

Hence, $(x_1 + x_2 + \dots + x_n, y)_{\text{i-s}} \leq \| y \| [\, \| x_1 \| + \| x_2 \| + \dots + \| x_{n-1} \| \,] + (x_n, y)_{\text{i-s}}$.

Corollary 3.2. [3] Let $x_1, x_2, y \in X$. Then

$$(x_1 + x_2, y)_{\text{i-s}} \leq \| y \| \| x_1 \| + (x_2, y)_{\text{i-s}}.$$

Theorem 3.3. Let $x_1, x_2, \dots x_n, y \in X$ and let $a_1, a_2, \dots a_n \in R$. Then

$$(a_1x_1 + a_2x_2 + \dots + a_nx_n + y, x_1 + x_2 + \dots + x_n)_{\text{i-s}}$$
$$= a_1\| x_1 \| + a_2\| x_2 \| + \dots + a_n\| x_n \| + (y, x_1 + x_2 + \dots + x_n)_{\text{i-s}}.$$

Proof. In [5], we can conclude that

$$(a_1x_1 + a_2x_2 + \dots + a_nx_n + y, x_1 + x_2 + \dots + x_n)_{\text{i}}$$
$$= a_1\| x_1 \| + a_2\| x_2 \| + \dots + a_n\| x_n \| + (y, x_1 + x_2 + \dots + x_n)_{\text{i}}$$

and

$$(a_1x_1 + a_2x_2 + \dots + a_nx_n + y, x_1 + x_2 + \dots + x_n)_{\text{s}}$$
$$= a_1\| x_1 \| + a_2\| x_2 \| + \dots + a_n\| x_n \| + (y, x_1 + x_2 + \dots + x_n)_{\text{s}}.$$

Thus, $2(a_1x_1 + a_2x_2 + \dots + a_nx_n + y, x_1 + x_2 + \dots + x_n)_{\text{i-s}}$

$$= (a_1x_1 + a_2x_2 + \dots + a_nx_n + y, x_1 + x_2 + \dots + x_n)_{\text{i}}$$
$$+ (a_1x_1 + a_2x_2 + \dots + a_nx_n + y, x_1 + x_2 + \dots + x_n)_{\text{s}}$$
$$= a_1\| x_1 \| + a_2\| x_2 \| + \dots + a_n\| x_n \| + (y, x_1 + x_2 + \dots + x_n)_{\text{i}}$$
$$+ a_1\| x_1 \| + a_2\| x_2 \| + \dots + a_n\| x_n \| + (y, x_1 + x_2 + \dots + x_n)_{\text{s}}$$
$$= 2[\, a_1\| x_1 \| + a_2\| x_2 \| + \dots + a_n\| x_n \| \,]$$
$$+ (y, x_1 + x_2 + \dots + x_n)_{\text{i}} + (y, x_1 + x_2 + \dots + x_n)_{\text{s}}$$
$$= 2[\, a_1\| x_1 \| + a_2\| x_2 \| + \dots + a_n\| x_n \| \,] + 2(y, x_1 + x_2 + \dots + x_n)_{\text{i-s}}.$$

Hence, $(a_1x_1 + a_2x_2 + \dots + a_nx_n + y, x_1 + x_2 + \dots + x_n)_{\text{i-s}}$

$$= a_1\| x_1 \| + a_2\| x_2 \| + \dots + a_n\| x_n \| + (y, x_1 + x_2 + \dots + x_n)_{\text{i-s}}.$$

Corollary 3.4. [3] Let $x, y \in X$ and let $a \in R$. Then

$$(ax + y, x)_{\text{i-s}} = a\| x \| + (y, x)_{\text{i-s}}.$$

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