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On the Diophantine Equation $24^x + 27^y = z^2$

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Abstract

In this paper, we show that (1, 0, 5) is a unique non-negative integer solution (x, y, z) for the Diophantine equa-tion $24^x + 27^y = z^2$ where x, y and z are non-negative inte-gers.

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1 Introduction

In 2007, Acu [1] showed that (3, 0, 3) and (2, 1, 3) are only two solutions (x, y, z) for the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers.
In 2011, Suvarnamani, Singta and Chotchaisthit [11] showed that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution where x, y and z are non-negative integers.
In 2012, Sroysang [5] showed that (1, 0, 2) is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers.
In 2013, Rabago [4] showed that (1, 0, 3), (1, 1, 5), (2, 1, 9) and (3, 1, 23) are only four solutions (x, y, z) for the Diophantine equa-tion $8^x + 17^y = z^2$ where x, y and z are non-negative integers.
In 2013, Chotchaisthit [2] showed that (3, 0, 3) is a unique non-negative integer solution (x, y, z) for the Diophantine equation $2^x + 11^y = z^2$ where x, y and z are non-negative integers.
In this paper, we show that (1, 0, 5) is a unique non-negative integer solution (x, y, z) for the Diophantine equation $24^x + 27^y = z^2$ where x, y and z are non-negative integers.

2 Preliminaries

Proposition 2. 1. [3] (Catalan's conjecture) (3, 2, 2, 3) is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$.

Lemma 2. 2. (1, 5) is a unique solution (x, z) for the Dio-phantine equation $24^x + 1 = z^2$ where x and z are non-negative integers.

Proof. Let x and z be non-negative integers such that $24^x + 1 = z^2$. We obtain that $x \geq 1$. Then $z^2 = 24^x + 1 \geq 24^1 + 1 = 25$. Thus, $z \geq 5$. Now, we consider on the equation $z^2 - 24^x = 1$. By Proposition 2. 1, we have x = 1. Then z = 5.

Lemma 2. 3. The Diophantine equation $1 + 27^y = z^2$ has no non-negative integer solution where y and z are non-negative integers.

Proof. Suppose that there are non-negative integers y and z such that $1 + 27^y = z^2$. We obtain that $y \geq 1$. Then $z^2 = 1+27^y \geq 1+27^1 = 28$. Thus, $z \geq 6$. Now, we consider on the equation $z^2 - 27^y = 1$. By Proposition 2. 1, we have y = 1. Then $z^2 = 28$, this is a contradiction.

3 Main Results

Theorem 3. 1. (1, 0, 5) is a unique non-negative integer solu-tion (x, y, z) for the Diophantine equation $24^x + 27^y = z^2$ where x, y and z are non-negative integers.

Proof. Let x, y and z be non-negative integers such that $24^x + 27^y = z^2$. By Lemma 2. 3, we have $x \geq 1$. It follows that z is odd. Then z = 2t + 1 for some a non-negative integer t. Thus, $24^x + 27^y = 4(t^2 + t) + 1$. This implies that $27^y \equiv 1 \pmod 4$. It follows that y is even. Then y = 2k for some a non-negative integer k. Now, we will divide the number y into two cases.

Case y = 0. By Lemma 2. 2, we obtain that x = 1 and z = 5. Case $y \geq 2$. Then $k \geq 1$. Then $z^2 - 27^{2k} = 24^x$. Then $(z - 27^k)(z + 27^k) = 24^x$. Thus, $z - 27^k = 2^u 3^w$ where u and w are non-negative integers. Then $z + 27^k = 2^{3x-u} 3^{x-w}$. Thus, $2(3^{3k}) = 2(27^k) = 2^{3x-u} 3^{x-w} - 2^u 3^w = 2(2^{3x-u-1} 3^{x-w} - 2^{u-1} 3^w)$. Then either $3x - u = 1$ or $u = 1$. Subcase $3x - u = 1$. Then $2(3^{3k}) = 2(3^{x-w}) - 2^{3x-1} 3^w = 2(3^{3k})(3^{x-w-3k} - 2^{3x-2} 3^{w-3k})$. Then $3^{x-w-3k} - 2^{3x-2} 3^{w-3k} = 1$. Then $w - 3k = 0$. This implies that $3^{x-6k} - 2^{3x-2} = 1$. Note that $3x - 2 > 0$ and then $x - 6k > 0$. Since $k \geq 1$, we have $x > 6k \geq 6$ and so $3x - 2 > 16$. If $x - 6k = 1$ then $3x - 2 = 1$, this is impossible. Thus, $x - 6k > 1$. By Proposition 2. 1, we have $x - 6k = 2$ and $3x - 2 = 3$. Thus, $x = {}^1$.

This is a contradiction.

Subcase u = 1. We obtain that $2(3^{3k}) = 2^{3x-1}3^{x-w} - 2(3^w) = 2(3^{3k})(2^{3x-2}3^{x-w-3k} - 3^{w-3k})$. Then $2^{3x-2}3^{x-w-3k} - 3^{w-3k} = 1$. If $w - 3k = 0$ then $x = 1$ and then $0 = x - w - 3k = 1 - 6k$, this is impossible. Then $x-w-3k = 0$. It follows that $2^{3x-2} - 3^{w-3k} = 1$. Note that $3x-2 > 0$ and $w-3k > 0$. If $3x-2 = 1$ then $w-3k = 0$, is impossible. have $w - 3k = 1$. contradiction.

Corollary 3. 2. Thus, $3x - 2 > 1$. By Proposition 2. 1, we Then $3x - 2 = 2$. Thus, $x = 3$. This is a The Diophantine equation $24^x + 27^y = w^4$ has no non-negative integer solution where x, y and w are non-negative integers.

Proof. Suppose that there are non-negative integers x, y and w such that $24^x + 27^y = w^4$. Let $z = w^2$. Then $24^x + 27^y = z^2$. By Theorem 3. 1, we have $(x, y, z) = (1, 0, 5)$. Then $w^2 = z = 5$. This is a contradiction.

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