Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 10, Number 2 (2014), pp. 159-162

http://www.ripublication.com

On a Question by Laurent Schwartz

Nuno C. Freire

CIMA DMAT Universidade de Évora
Col. Verney R. Romão Ramalho 59 7000 Évora Portugal
AND Mailing Address R. Vicente Arnoso, nº 80, 3º
2765-459 São João do Estoril T Portugal
E: mail freirenuno2003@iol.pt

ABSTRACT

We prove that each function f in C (F, T) where, T is a closed interval in an infinite Banach space with an unconditional Schauder basis and F is a closed subspace of a Hausdorff topological space having a normal neighborhood has a continuous extension on E to T. This way we obtain one answer to a question by Laurent Schwartz, his book in general topology and functional analysis that we quote in the references.

AMS Subject Classification (2000) Primary 54D15, 46B05

Key words and phrases normal, extension, continuous

This work is in its final form and no version of it will be submitted for publication elsewhere.

INTRODUCTION

Following [3] Corollaire 2 to Théorème (T.2, XXII, 2;2) (pp. 347, 350)we have the extension property (Rfin) Consider the extended real line $[-\infty,+\infty]$ equipped with the usual topology generated by the open intervals of **R** and the intervals $[-\infty,b)$, $(a,+\infty]$ (a, b reals) and, let T be a subspace of the form (I, U) where I is an interval and U is the induced topology by the usual topology of the real line. Or let T stand for a finite dimensional Banach space G. For $f : F \subset E \to T$ a continuous function where, F is a closed subspace of the Hausdorff topological space E having a normal, closed neighborhood, there is a continuous extension of f on E to T. Laurent Schwartz notices, Remarque 2. P. 352 in [3] that the corresponding result for T a closed ball in an infinite dimensional Banach space X (or $T = X$) possibly holds, which we believe has remained as an open problem. We prove in Paragraph 3., Extension Result that, we may take for X a Banach space with an unconditional Schauder basis. Also, possibly a closed interval in X or a closed ball in X . In the Preliminaries, we set the background. The scalars are assumed to be real.

PRELIMINARIES

Recall ([2]) that we say the infinite dimensional Banach space $(X,\| \|)$ has an unconditional Schauder basis (a basis) (e_n) if (e_n) is a sequence of linearly independent vectors in X such that, each vector x in the space has a representation $x=\sum_{n=1}^{\infty}\lambda_n e_n = \lim_{N\to\infty}\sum_{n=1}^{N}\lambda_n e_n$ in the sense that $\|x-(\lambda_1 e_1+\lambda_2 e_2+...+\lambda_n e_n)\| \to_{n\to\infty} 0$. Here, the scalars λ_n are unique, depending on x and further it is required that the series $\sum_n \lambda_{\pi(n)} e_{\pi(n)} = x$ for each bijection $\pi : \mathrm{N}\to\mathrm{N}$.

Remark 2. 1. We may suppose that the basis (e_n) is determined by norm 1 vectors and has basis constant equal to one that is, the inequality $\|\lambda_1 e_1+...+\lambda_{n(1)}e_{n(1)}\| \le \|\lambda_1 e_1+...+\lambda_{n(2)}e_{n(2)}\|$ for each choice of reals λ_n, $1\le n(1)\le n(2)$ holds. This follows from Theorem. 4.1.24, p. 358 in [2].

Also recall the Bounded Multiplier Test ([2], 4.2.8, p. 371) stating that the series $\sum_n \lambda_n e_n = x$ converges if and only if each series $\sum_n \alpha_n\lambda_n e_n$ is convergent, whatever be the bounded sequence $(\alpha_n)\in l_\infty$. We may also find in [2], after the Author showing that (Lemma 4.2.7) for $x=\sum_n \lambda_n e_n$ there is a constant M such that $\sup_n\|\alpha_1\lambda_1 e_1+...+\alpha_n\lambda_n e_n\| \le M\|(\alpha_n)_\infty\|$ where $\|(\alpha_n)_\infty\| = \sup\{|\alpha_n : n=1,2,...\}|$ the following

Lemma 2.2. Putting $\|\lambda_1 e_1+...+\lambda_n e_n+...\|_{bmu} = \sup\{\|\alpha_1\lambda_1 e_1+...+\alpha_n\lambda_n e_n+...\| : \|(\alpha_n\|_\infty \le 1\}$ it holds that the bmu-norm $\|\lambda_1 e_1+...+\lambda_n e_n+...\|_{bmu}$ is an equivalent norm to the original norm of X . Thee is a constant $C\succ 0$ such that $\|x\| \le \|x\|_{bmu} \le C\|x\|$.

Proof: This follows from Theorem. 4.2.16, p. 373. See also Definition. 4. 1.12 and Theorem. 4.1.14, p. 354.

Lemma 2.3. The bmu-norm satisfies that $\|\lambda_1 e_1+...+\lambda_n e_n+...\|_{bmu} \le \|\mu_1 e_1+...+\mu_n e_n+...\|_{bmu}$ whenever $\lambda_1 e_1+...+\lambda_n e_n+...,\mu_1 e_1+...+\mu_n e_n+...\in X, |\lambda_1| \le |\mu_1|,...,|\lambda_n| \le \|\mu_n|,...$

Remark 2.4. In the above Lemma 2.3. we have that the convergence of $\mu_1 e_1 + ... + \mu_n e_n + ...$ implies the convergence of $\lambda_1 e_1 + ... + \lambda_n e_n + ...$

Proof: WE find that $\|\lambda_m e_m + ... + \lambda_{m+p} \|_{bmu} \leq \|\mu_m + ... + \mu_{m+p} \|_{bmu} \to_{m,p\to\infty} 0$ by Lemma 2.3. and the result follows.

Recall ([2]) the associate linear functionals $< e_m*, \sum_n \lambda_n e_n >= \lambda_m$ which are continuous. It turns out that for $f : F \subset E \to X$ a continuous function where E is a Hausdorff topological space we have $f(u) = \sum_n < e_n*, f(u) >= \sum_n f_n(u) e_n$, $f_n = e_n * of : F \subset E \to R$.

Extension Result

Notation 1. Following [2], we put $a \leq b$ for $a = \sum_n \alpha_n e_n, b = \sum_n \beta_n e_n$ meaning that $\alpha_n \leq \beta_n, n = 1,2,...$

Notation 2. Letting $a \leq b$ as above, we put $[a,b] = \{\sum_n \lambda_n e_n : \alpha_n \leq \lambda_n \leq \beta_n, n = 1,2,...\}$ and we say that $[a,b]$ is a closed interval in X .

Theorem 1. Let $T = [a,b]$ be a closed interval in the infinite dimensional Banach space X with the unconditional Schauder basis (e_n) as in the Preliminaries. For each continuous function $f : F \subset E \to T$ where F is a closed subspace of E having a normal neighborhood, there is a continuous extension $\hat{f} : E \to T$.

Proof: WE let $f(u) = \sum_k f_k(u) e_k \quad (u \in F)$ where $f_k = e_k * of : F \subset E \to [\alpha_k, \beta_k]$, $f(u) = x(u) = \sum_k \lambda_k e_k, \alpha_k \leq \lambda_k \leq \beta_k$, $f_k(u) = \lambda_k(u) = \lambda_k$. Following (Rfin) as inn the Introduction, there is a continuous extension $\hat{f}_k : E \to [\alpha_k, \beta_k]$ of f_k. We find that $\alpha_k \leq \hat{f}_k(u) \leq \beta_k$ .The series $\sum_k \alpha_k e_k = a, \sum_k \beta_k e_k = b, \sum_k |\alpha_k \ | e_k = |a \ |, \sum_k |\beta_k \ | e_k = |b \ |$ converge (apply the equivalent bmu-norm), $|\lambda_k \ | \leq \max\{|\alpha_k \ |, \beta_k \ |\}$. Apllying Remark 2.4., we see that the series $\sum_k \hat{f}_k(u) e_k = \hat{f}(u)$ is convergent. Now we have obtained a continuous function $\hat{f} : E \to T$ that is an extension of f. In fact, it holds that $\sup\{|\hat{f}_n(u)| : u \in E\} \leq \max\{|\alpha_n \ |, |\beta_n \ |\} \to_{n\to\infty} 0$ hence letting $n \geq N$ imply that $\sup\{\hat{f}_n(u) - \hat{f}_n(x) \ | \leq \varepsilon$, we may find a neighborhood V of x such that

$\sup\{\left|\hat{f}_n(u) - \hat{f}_n(x)\right| : n = 1,2,...\} \le \varepsilon$ which implies that $\left\|\hat{f}(u) - \hat{f}(x)\right\| \le \varepsilon$ for all u in V . The proof is complete

REFERENCES

[1] DUGUNDJI, JAMES Topology Allyn and Bacon, Inc. 470 Atlantic Avenue, Boston (1966)

[2] MEGGINSON, ROBERT E. An Introduction to Banach Space Theory, Graduate Texts in Mathematics 183, Springer New York Berlin Heidelberg Barcelona Budapest Hong Kong London Milan Paris Singapore Tokyo (1998)

[3] SCHWARTZ, LAURENT Analyse Deuxième Partie Topologie générale et analyse fonctionnelle Hermann 156, boulevard Saint-Gerrmain, Paris VI (1970).