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On Ideal Convergent of Double Sequences in p-Adic Linear 2-Normed Spaces

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ABSTRACT

In this paper, we introduce I_2-limit operation for double sequences in p-adic linear 2-normed space ($X, N(\bullet,\bullet)_p$) is linear with respect to summation and scalar multiplication and we investigate the relation between I_2-limit points and I_2-cluster points of p-adic linear 2-normed spaces.

Keywords: p-adic linear 2-normed space, double sequence, statistically convergent, admissible ideal.

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1. Introduction:

The idea of $I-$convergence is based on the notion of the ideal I of subsets of N, the set of natural numbers. The notion of ideal convergence for single sequences was introduced first by P.Kostyrko et al [12,13] as an interesting generalization of statistical convergence. F.Nuray and Ruckle [19] independently introduced the same concept as the name generalized statistical convergence.

The concept of a double sequence was initially introduced by Pringsheim [21] in the 1900s and this concept has been studied by many others. A double sequence of real (complex) numbers is a function from $N \times N$ to F (where F= R or C) and we denote a double sequence as (x_{ij}) where the two subscripts run through the sequence of Natural numbers independent of each other. Das et al [5] introduced the concept of $I-$convergence of double sequences in a metric space and studied some properties.

Also, Pratulananda Das and Prasanta Malik [20] defined the concept of $I-$limit points, $I-$cluster points, $I-$limit superior and $I-$limit inferior of double sequences. B.Tripathy, B.C.Tripathy [4] introduced the notion of $I-$convergence and $I-$Cauchy sequence for double sequences and also Vijay Kumar [27] discussed the basic properties of I and I^*-convergence for double sequences.

The concept of linear 2-normed spaces has been investigated by Gähler in 1965 [6] and has been developed extensively in different subjects by others [7, 9, 10, 11, 22]. Lewandowska published a series of papers on 2-normed sets and generalized 2-normed spaces, convergent sequences, 2-Banach spaces, etc., (see [15], [16], [17], for more details). A.Sahiner et al [24] introduced $I-$cluster points of convergent sequences in 2-normed linear spaces and Gurdal [8] investigated the relation between $I-$cluster points and ordinary limit points of sequences in 2-normed spaces. The concept of $I-$convergence for the double sequences in 2-normed spaces introduced by Saeed Sarabadan and Sorayya Talebi [23].

Mehmet Acikgoz [18] introduced a very understandable and readable connection between the concepts in p-adic numbers, p-adic analysis and linear 2-normed spaces. B.Surender Reddy [25] introduced some properties of p-adic linear 2-normed spaces and obtained necessary and sufficient conditions for p-adic 2-norms to be equivalent on p-adic linear 2-normed spaces. Recently B.Surender Reddy and D.Shankaraiah [26] introduced $I-$convergence of sequences and their properties in p-adic linear 2-normed spaces.

The main aim of this paper is we introduce I_2-limit operation for double sequences in p-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is linear with respect to summation and scalar multiplication and we investigate the relation between I_2-limit points and I_2-cluster points of p-adic linear 2-normed spaces.

2. Preliminaries:

In this paper, we will use the notations; p for a prime number, Z - the ring of rational integers, Z^+ - the positive integers, Q - the field of rational numbers, R - the field of real numbers, R^+ - the positive real numbers, Z_p - the ring of p-adic rational integers, Q_p - the field of p-adic rational numbers, C - the field of complex numbers and C_p - the p-adic completion of the algebraic closure of Q_p.

Definition 2.1: A double sequence $x = (x_{ij})$ is said to be convergent to a number ξ in the Pringsheim's sense if for each $\varepsilon > 0$ there exists a positive integer m such that $|x_{ij} - \xi| < \varepsilon$ whenever $i, j \geq m$. Then the number ξ is called the Pringsheim limit of the sequence x and we write as $P - \lim_{i,j\to\infty} x_{ij} = \xi$.

Definition 2.2: A double sequence $x=(x_{ij})$ is said to be Cauchy sequence if for each $\varepsilon>0$ there exists a positive integer n_0 such that $|x_{ij}-x_{mn}|<\varepsilon$ for every $i\geq m\geq n_0$ and $j\geq n\geq n_0$.

Definition 2.3: A double sequence $x=(x_{ij})$ is said to be bounded if there exists a real number $M>0$ such that $|x_{ij}|<M$ for each i and j.

Definition 2.4: Let $K\subset N\times N$ and $K(m,n)=\{(i,j):(i,j)\in K; i\leq m, j\leq n\}$.If the sequence $\left\{\frac{K(m,n)}{mn}\right\}$ has a limit in Pringsheim's sense then we say that K has a double natural density and it is denoted as $\lim_{m,n\to\infty}\frac{K(m,n)}{mn}=\delta_2(K)$.

Definition 2.5: A double sequence $x=(x_{ij})$ is said to be statistically convergent to a number ξ if for each $\varepsilon>0$, the set $A(\varepsilon)=\{(i,j)\in N\times N:|x_{ij}-\xi|\geq\varepsilon\}$ has double natural density zero. If $x=(x_{ij})$ is statistically convergent to ξ then we write $St-\lim_{i,j\to\infty}x_{ij}=\xi$.

Definition 2.6: Let I_2 be an ideal in $N\times N$. A double sequence $x=(x_{ij})$ is said to be I_2-convergent to L in Pringsheim's sense if for each $\varepsilon>0$, the set $\{(i,j)\in N\times N:|x_{ij}-L|\geq\varepsilon\}\in I_2$ and L is called I_2-limit of $x=(x_{ij})$ and we write $I_2-\lim_{i,j\to\infty}x_{ij}=L$.

Definition 2.7: A double sequence $x=(x_{ij})$ is said to be I_2^*-convergent to ξ if there exists a set $M=\{(i,j):i,j=1,2,3,.......\}\in F(I_2)$ $(i.e,(N\times N)-M\in I_2)$ such that $\lim_{i,j\to\infty}x_{ij}=\xi$ and ξ is called I_2^*-limit of $x=(x_{ij})$ and we write $I_2^*-\lim x_{ij}=\xi$.

Definition 2.8: A double sequence $x=(x_{ij})$ is I_2-convergent to zero in Pringsheim's sense is called I_2-null double sequence in Pringsheim's sense.

Suppose a mapping $d_p: X\times X\times X\to R$ on a non-empty set X satisfying the following conditions, for all $x,y,z\in X$

D_1) For any two different elements x and y in X there is an element z in X such that $d_p(x,y,z)\neq 0$

D_2) $d_p(x,y,z)=0$ when two of three elements are equal

D_3) $d_p(x,y,z)=d_p(x,z,y)=d_p(y,z,x)$

D_4) $d_p(x,y,z)\leq d_p(x,y,w)+d_p(x,w,z)+d_p(w,y,z)$ for any w in X. Then d_p

is called p-adic 2-metric on X and the pair (X, d_p) is called p-adic 2-metric space. If p-adic 2-metric also satisfies the condition

$$d_p(x,y,z) \le \max\{d_p(x,y,w), d_p(x,w,z), d_p(y,w,z)\} \quad \text{for } x,y,z,w \in X,$$

then d_p is called a p-adic ultra 2-metric and the pair (X, d_p) is called a p-adic ultra 2-metric space.

Definition 2.9: Let X be a linear space of dimension greater than 1 over K, where K is the real or complex numbers field. Suppose $N(\bullet,\bullet)_p$ be a non-negative real valued function on $X \times X$ satisfying the following conditions:

$(2-pN_1)$: $N(x,z)_p = 0$ if and only if x and z are linearly dependent vectors.

$(2-pN_2)$: $N(xy,z)_p = N(x,z)_p . N(y,z)_p$ for all $x,y,z \in X$,

$(2-pN_3)$: $N(x+y,z)_p \le N(x,z)_p + N(y,z)_p$ for all $x,y,z \in X$,

$(2-pN_4)$: $N(\lambda x,z)_p = |\lambda|\, N(x,z)_p$ for all $\lambda \in K$ and $x,z \in X$.

Then $N(\bullet,\bullet)_p$ is called a p-adic 2-norm on X and the pair $(X, N(\bullet,\bullet)_p)$ is called p-adic linear 2-normed space.

For every p-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ the function defined on $X \times X \times X$ by $d_p(x,y,z) = N(x-z, y-z)_p$ is a p-adic 2-metric. Thus every p-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ will be considered to be a p-adic 2-metric space with this 2-metric. A double sequence $\{x_{ij}\}$ of p-adic 2-metric space (X, d_p) converges to $x \in X$ if for every $\varepsilon > 0$, there is an $l \ge 1$ such that $d_p(x_{ij}, x, z) = N(x_{ij} - z, x - z)_p < \varepsilon$ for every $i, j \ge l$. For the given two double sequences of p-adic 2-metric space (X, d_p) which are $\{x_{ij}\}$ and $\{y_{ij}\}$ converges to $x, y \in X$ in the p-adic 2-metric space respectively, then the double sequence of sums $x_{ij} + y_{ij}$ and the product $x_{ij} y_{ij}$ converges to the sum $x + y$ and to the product xy of the limits of initial double sequences.

A double sequence $\{x_{ij}\}$ of p-adic 2-metric space (X, d_p) is a Cauchy sequence with respect to the p-adic 2-metric if for each $\varepsilon > 0$, there is an $l \ge 1$ such that $d_p(x_{ij}, x_{mn}, z) = N(x_{ij} - z, x_{mn} - z)_p < \varepsilon$, for every $i \ge m \ge l$, $j \ge n \ge l$.

Definition 2.10: A double sequence $x = (x_{ij})$ in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is said to be convergent to $l \in X$ if for each $\varepsilon > 0$ there exists $m \in N$ such that $N(x_{ij} - l, z)_p < \varepsilon$ for each $i, j \ge m$ and for each $z \in X$. If $x = (x_{ij})$ is convergent to l then we write $\lim_{i,j\to\infty} x_{ij} = l$ or $x_{ij} \xrightarrow{[N(\bullet,\bullet)_p]_X} l$.

Definition 2.11: A double sequence $x = (x_{ij})$ in a $p-$adic linear 2-normed space

$(X, N(\bullet,\bullet)_p)$ is said to be bounded if for each non zero $z \in X$ and for all $i, j \in N$ there exists $M > 0$ such that $N(x_{ij}, z)_p < M$. Note that a convergent double sequence need not be bounded.

Definition 2.12: A double sequence $x = (x_{ij})$ in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is said to be Cauchy sequence if for each $\varepsilon > 0$ there exists a positive integer n_0 such that $N(x_{ij} - x_{mn}, z)_p < \varepsilon$ for every $i \geq m \geq n_0$ and $j \geq n \geq n_0$.

A p-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is called complete if every Cauchy sequence is convergent in p-adic linear 2-normed space. A p-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is called p-adic 2-Banach space if p-adic linear 2-normed space is complete.

Proposition 2.13: If a double sequence $\{x_{ij}\}$ in a p-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is convergent to $x \in X$, then $\lim_{i,j\to\infty} N(x_{ij}, z)_p = N(x, z)_p$ for each z∈ X.

Proposition 2.14: If $\lim_{i,j\to\infty} N(x_{ij}, z)_p$ exists then we say that $\{x_{ij}\}$ is a Cauchy sequence with respect to $N(\bullet,\bullet)_p$.

Proof: Let us suppose that $\lim_{i,j\to\infty} N(x_{ij}, z)_p = x$. Then we can obtain a constant M_1 such that $i, j > M_1 \Rightarrow N(x - x_{ij}, z)_p < \frac{\varepsilon}{2}$. If $i, j, m, n > M_1$ then $N(x - x_{ij}, z)_p < \frac{\varepsilon}{2}$ and $N(x - x_{mn}, z)_p < \frac{\varepsilon}{2}$, hence by using the triangle inequality, we have

$$N(x_{ij} - x_{mn}, z)_p = N(x_{ij} - x + x - x_{mn}, z)_p \leq N(x_{ij} - x, z)_p + N(x - x_{mn}, z)_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow \{x_{ij}\}$ is a Cauchy sequence with respect to $N(\bullet,\bullet)_p$.

3. Main Results:

In this section, we prove that I_2-limit operation for double sequences in p-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is linear with respect to summation and scalar multiplication. We investigate the relation between I_2-cluster points and I_2-limit points of double sequences in $p-$adic linear 2-normed spaces.

A family of sets $I \subseteq 2^Y$ (power sets of Y) is said to be an ideal if $\Phi \in I$, I is additive i.e., $A, B \in I \Rightarrow A \bigcup B \in I$ and hereditary i.e., $A \in I$, $B \subseteq A \Rightarrow B \in I$.

A non empty family of sets $F \subset 2^Y$ is a filter on Y if and only if $\Phi \notin F$, $A \bigcap B \in F$ for each $A, B \in F$, and any subset of an element of F is in F . An ideal I

is called non-trivial if $I \neq \Phi$ and $Y \notin I$. Clearly I is a non-trivial ideal if and only if $F = F(I) = \{Y - A : A \in I\}$ is a filter in Y, called the filter associated with the ideal I. A non-trivial ideal I is called admissible if and only if $\{\{n\} : n \in Y\} \subset I$.

An admissible ideal $I \subset 2^Y$ is said to have the property (AP) if for any sequence $\{A_1, A_2, A_3, ...\}$ of mutually disjoint sets of I there is a sequence $\{B_1, B_2, B_3, ...\}$ of sets such that each symmetric difference $A_i \Delta B_i$, $i = 1,2,3....,$ is finite and $B = \bigcup_{i=1}^{\infty} B_i \in I$.

In order to distinguish between the ideals of N and $N \times N$ we shall denote the ideals of N by I and ideals of $N \times N$ by I_2. In general, there is no connection between I and I_2.

A non trivial ideal I_2 in $N \times N$ is called strongly admissible if $\{i\} \times N$ and $N \times \{i\}$ belong to I_2 for each $i \in N$. It is clear that a strongly admissible ideal is admissible also.

Let $I_0 = \{A \subset N \times N : (\exists m(A) \in N)(i, j \geq m(A) \Rightarrow (i, j) \in (N \times N) - A\}$. Then I_0 is a non trivial strongly admissible ideal and I_2 is strongly admissible if and only if $I_0 \subseteq I_2$. $I_2 \subset 2^{N \times N}$ is a non trivial ideal if and only if the class $F = F(I) = \{(N \times N) - A : A \in I_2\}$ is a filter in $N \times N$.

Now introducing the definition of I_2 – convergence for double sequence $x = (x_{ij})$ in a p – adic linear 2-normed space as follows.

Definition 3.1: A double sequence $x = (x_{ij})$ in a p – adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is said to be I_2 – convergent to $l \in X$ if for each $\varepsilon > 0$ and non zero $z \in X$, the set $A(\varepsilon) = \{(i, j) \in N \times N : N(x_{ij} - l, z)_p \geq \varepsilon\} \in I_2$ and l is called the I_2 – limit of the sequence $x = (x_{ij})$.

If $x = (x_{ij})$ is I_2 – convergent to l, then we write $I_2 - \lim_{i,j \to \infty} x_{ij} = l$ or $I_2 - \lim_{i,j \to \infty} N(x_{ij} - l, z)_p = 0$ or $I_2 - \lim_{i,j \to \infty} N(x_{ij}, z)_p = N(l, z)_p$ for each non zero $z \in X$.

Corollary 3.2: I_2 – limit of convergent double sequence in a p – adic linear 2-normed space is unique.

Proof: Let $x = (x_{ij})$ be a convergent double sequence in a p – adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$. If possible suppose that l_1, l_2 are two distinct I_2 – limits of $x = (x_{ij})$ in X.

Since $l_1 \neq l_2$, there exists $z \in X$ such that $l_1 - l_2$ and z are linearly independent. Put $N(l_1 - l_2, z)_p = 2\varepsilon$ for $\varepsilon > 0$. Again since l_1, l_2 are I_2 – limits of $x = (x_{ij})$ in X, therefore by definition of I_2 – convergence we have

$$A_1(\varepsilon)=\{(i,j)\in N\times N : N(x_{ij}-l_1,z)_p\geq\varepsilon\}\in I_2$$

and $\quad A_2(\varepsilon)=\{(i,j)\in N\times N : N(x_{ij}-l_2,z)_p\geq\varepsilon\}\in I_2$, for each non zero $z\in X$.

Now

$$\begin{aligned}2\varepsilon &= N(l_1-l_2,z)_p\\ &= N(l_1-x_{ij}+x_{ij}-l_2,z)_p\\ &= N((x_{ij}-l_2)-(x_{ij}-l_1),z)_p\\ &\leq N(x_{ij}-l_1,z)_p+N(x_{ij}-l_2,z)_p \qquad (3.3)\end{aligned}$$

Let $(i,j)\in A_2^c$. Then $N(x_{ij}-l_2,z)_p<\varepsilon$ (3.4)

From equation (3.3) and equation (3.4), $N(x_{ij}-l_1,z)_p\geq\varepsilon$ and hence $(i,j)\in A_1(\varepsilon)$.

$$\Rightarrow A_2^C\subset A_1(\varepsilon)\in I_2$$

$$\Rightarrow A_2^c=\{(i,j)\in N\times N : N(x_{ij}-l_2,z)_p<\varepsilon\}\in I_2$$

Since $A_2, A_2^c\in I_2$, therefore $A_2\cup A_2^c=N\times N\in I_2$ which contradict with non trivial I_2. Thus our assumption that l_1, l_2 are distinct is wrong and hence I_2 – limit of convergent double sequence in a p – adic linear 2-normed space is unique.

Corollary 3.5: Let $x=(x_{ij})$ and $y=(y_{ij})$ be two double sequences in a p – adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ and $I_2-\lim_{i,j\to\infty} x_{ij}=l_1, I_2-\lim_{i,j\to\infty} y_{ij}=l_2$. Then

(a). $I_2-\lim_{i,j\to\infty}(x_{ij}+y_{ij})=l_1+l_2$

(b). $I_2-\lim_{i,j\to\infty}\alpha x_{ij}=\alpha l_1$, where $\alpha\in R$

Proof: (a) Let $x=(x_{ij})$, $y=(y_{ij})$ be two double sequences in a p – adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ and $I_2-\lim_{i,j\to\infty} x_{ij}=l_1, I_2-\lim_{i,j\to\infty} y_{ij}=l_2$. Then by definition of I_2 – convergence, for each $\varepsilon>0$,

$$A_1=A_1\left(\frac{\varepsilon}{2}\right)=\{(i,j)\in N\times N : N(x_{ij}-l_1,z)_p\geq\frac{\varepsilon}{2}\}\in I_2$$ and

$$A_2=A_2\left(\frac{\varepsilon}{2}\right)=\{(i,j)\in N\times N : N(x_{ij}-l_2,z)_p\geq\frac{\varepsilon}{2}\}\in I_2$$, for each non zero $z\in X$.

Let $A=A(\varepsilon)=\{(i,j)\in N\times N : N(x_{ij}+y_{ij}-(l_1+l_2),z)_p\geq\varepsilon\}$, for each non zero $z\in X$.

Suppose $(i,j)\in(A_1\cup A_2)^c$. Then $(i,j)\in A_1^c\cap A_2^c \Rightarrow (i,j)\in A_1^c$ and $(i,j)\in A_2^c$

$$\Rightarrow N(x_{ij}-l_1,z)_p<\frac{\varepsilon}{2} \text{ and } N(y_{ij}-l_2,z)_p<\frac{\varepsilon}{2}.$$

Now

$$\begin{aligned}N((x_{ij}+y_{ij})-(l_1+l_2),z)_p &= N((x_{ij}-l_1)+(y_{ij}-l_2),z)_p\\ &\leq N(x_{ij}-l_1,z)_p+N(y_{ij}-l_2,z)_p\end{aligned}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow (i, j) \in A^c$$

Therefore $(A_1 \cup A_2)^c \subset A^c \ \Rightarrow A \subset A_1 \cup A_2$

Since $A_1, A_2 \in I_2$, therefore $A_1 \cup A_2 \in I_2$ and hence $A \in I_2$ for each non zero $z \in X$. Thus $I_2 - \lim_{i,j\to\infty}(x_{ij} + y_{ij}) = l_1 + l_2 = I_2 - \lim_{i,j\to\infty} x_{ij} + I_2 - \lim_{i,j\to\infty} y_{ij}$ .i.e., $I_2 -$ limit operation for double sequences in $p-$ adic linear 2-normed space is linear with respect to summation.

(b). Let $I_2 - \lim_{i,j\to\infty} x_{ij} = l_1$ and $0 \neq \alpha \in R$. Then for each $\varepsilon > 0$, the set $\{(i, j) \in N \times N : N(x_{ij} - l_1, z)_p \geq \frac{\varepsilon}{|\alpha|}\} \in I_2$, for each non zero $z \in X$.

$\Rightarrow \{(i, j) \in N \times N :|\alpha| N(x_{ij} - l_1, z)_p \geq \varepsilon\} \in I_2$, for each non zero $z \in X$.

$\Rightarrow \{(i, j) \in N \times N : N(\alpha x_{ij} - \alpha l_1, z)_p \geq \varepsilon\} \in I_2$, for each non zero $z \in X$.

$\Rightarrow I_2 - \lim_{i.j\to\infty} \alpha x_{ij} = \alpha l_1$.

Thus $I_2 -$ limit operation for double sequences in $p-$ adic linear 2-normed space is linear with respect to scalar multiplication.

Definition 3.6: Let $K \subset N \times N$ such that for each $(i, j) \in N \times N$ there exists $(m,n) \in K$ such that $(m,n) > (i, j)$ with respect to the dictionary ordering. The sequence $(x)_K = \{x_{mn} : (m,n) \in K\}$ is said to be a subsequence of $x = (x_{ij})$ if $x = (x_{ij})$ is a double sequence in a $p-$ adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$. Otherwise If $x = (x_{ij})$ is a double sequence in a $p-$ adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ and K is a sub set of $N \times N$ such that for each $(i, j) \in N \times N$ there exists $(m,n) \in K$ such that $(m,n) > (i, j)$ with respect to the dictionary ordering, then $(x)_K = \{x_{mn} : (m,n) \in K\}$ is said to be a subsequence of $x = (x_{ij})$.

Definition 3.7: Let $x = (x_{ij})$ be a double sequence in a $p-$ adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$. An element $l \in X$ is said to be limit point of $x = (x_{ij})$ if there exists a sub sequence of $x = (x_{ij})$ which is convergent to l. The set of all limit points of a double sequence $x = (x_{ij})$ in X denoted by L_x^2.

Definition 3.8: Let $x = (x_{ij})$ be a double sequence in a $p-$ adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$. An element $l \in X$ is said to be $I_2 -$ limit point of $x = (x_{ij})$ if there exists a set $M = \{(m_i, m_j) : i, j \in N\} \subset N \times N$ such that $M \notin I_2$ and $\lim_{m_i,m_j\to\infty} x_{m_i m_j} = l$ and we denote the set of all $I_2 -$ limit points of $x = (x_{ij})$ by $I_2(\Lambda_x^2)$.

Definition 3.9: Let $x=(x_{ij})$ be a double sequence in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$. An element $l \in X$ is said to be I_2-cluster point of $x=(x_{ij})$ in X if for each $\varepsilon > 0$ the set $\{(i,j) \in N \times N : N(x_{ij}-l, z)_p < \varepsilon\} \notin I_2$ for each non zero $z \in X$ and we denote the set of all I_2-cluster points of $x=(x_{ij})$ by $I_2(\Gamma_x^2)$.

Theorem 3.10: Let I_2 be a strongly admissible ideal and $x=(x_{ij})$ be a double sequence in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$. Then $I_2(\Lambda_x^2) \subseteq I_2(\Gamma_x^2)$.

Proof: Let $\alpha \in I_2(\Lambda_x^2)$. This means α is a I_2-limit point of a double sequence $x=(x_{ij})$ in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$. Then there exists a set $M=\{(m_i, m_j) : i, j \in N\} \subset N \times N$ such that $M \notin I_2$ and

$$\lim_{m_i, m_j \to \infty} x_{m_i m_j} = \alpha \tag{3.11}$$

Let $\varepsilon > 0$. Then by equation (3.11) there exists $n_0 \in N$ such that for each $m_i, m_j \geq n_0$ we have $N(x_{m_i m_j}-\alpha, z)_p < \varepsilon$, for each $z \in X$.

$\Rightarrow \{(i,j) \in N \times N : N(x_{ij}-\alpha, z)_p < \varepsilon\} \supseteq M - \{(m_i, m_j) : \text{either } m_j \leq n_0 - 1 \text{ or } m_k \leq n_0 - 1\}$, for each $z \in X$.

Since I_2 is strongly admissible, therefore the set

$\{(i,j) \in N \times N : N(x_{ij}-\alpha, z)_p < \varepsilon\} \notin I_2$, for each $z \in X$.

$\Rightarrow \alpha$ is a I_2-cluster point of a double sequence $x=(x_{ij})$ in X.

$\Rightarrow \alpha \in I_2(\Gamma_x^2)$

Thus $I_2(\Lambda_x^2) \subseteq I_2(\Gamma_x^2)$.

Corollary 3.12: Let $(X, N(\bullet,\bullet)_p)$ be a finite dimensional $p-$adic linear 2-normed space and $I_2 \subset 2^{N \times N}$ be a admissible ideal. Then for each double sequence $x=(x_{ij})$ in X, $I_2(\Gamma_x^2)$ is closed in X.

Proof: Obviously $I_2(\Gamma_x^2) \subseteq \overline{I_2(\Gamma_x^2)}$ (3.13)

Let $\alpha \in \overline{I_2(\Gamma_x^2)}$. Then for each $\varepsilon > 0$, there exists, $l \in I_2(\Gamma_x^2) \cap B_u(\alpha, \varepsilon)$ where u is a basis for X. Choose $\delta > 0$ such that $B_u(l, \delta) \subseteq B_u(\alpha, \varepsilon)$. Obviously

$\{(m,n) \in N \times N : N(\alpha - x_{mn}, z)_p < \varepsilon\} \supseteq \{(m,n) \in N \times N : N(l - x_{mn}, z)_p < \delta\} \notin I_2$.

Therefore $\{(m,n) \in N \times N : N(\alpha - x_{mn}, z)_p < \varepsilon\} \notin I_2$ and $\alpha \in I_2(\Gamma_x^2)$.

$$\Rightarrow \overline{I_2(\Gamma_x^2)} \subseteq I_2(\Gamma_x^2) \tag{3.14}$$

From equation (3.13) and equation (3.14), $\overline{I_2(\Gamma_x^2)} = I_2(\Gamma_x^2)$ and hence $I_2(\Gamma_x^2)$ is closed in X.

Definition 3.15: Let $I_2 \subset 2^{N\times N}$ be an admissible ideal and $x=(x_{ij})$ be a double sequence in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$. If $K=\{(m,n): m,n\in N\}\in I_2$, then the sub sequence $x_K=(x_{mn})$ is called I_2-thin subsequence of double sequence $x=(x_{ij})$. If $M=\{(m,n): m,n\in N\}\notin I_2$, then the subsequence $x_M=(x_{mn})$ is called I_2-non thin subsequence of double sequence $x=(x_{ij})$.

Theorem 3.16: Let $I_2 \subset 2^{N\times N}$ be an admissible ideal and $x=(x_{ij})$, $y=(y_{ij})$ are double sequence in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ such that $M=\{(m,n)\in N\times N: x_{mn}\neq y_{mn}\}\in I_2$, then $I_2(\Lambda_x^2)=I_2(\Lambda_y^2)$ and $I_2(\Gamma_x^2)=I_2(\Gamma_y^2)$.

Proof: Let $\alpha\in I_2(\Lambda_x^2)$. Then there is a set $K=\{(k,l): k,l\in N\}\notin I_2$ such that $\lim_{k,l\to\infty} x_{kl}=\alpha$. Let $K_1=\{(m,n): m,n\in K, x_{mn}\neq y_{mn}\}$. Then $K_1\subset M$ and hence $K_1\in I_2$. Let $K_2=\{(m,n): m,n\in K, x_{mn}=y_{mn}\}$. Then $K_2\notin I_2$ (If $K_2\in I_2$ then $K=K_1\bigcup K_2\in I_2$, but $K\notin I_2$). Now the sequence $y_{K_2}=(y_{mn})$ is a I_2-non thin subsequence of double sequence $y=(y_{ij})$ and y_{K_2} converges to α in X. This implies that $\alpha\in I_2(\Lambda_y^2)$ and hence $I_2(\Lambda_x^2)\subseteq I_2(\Lambda_y^2)$ (3.17)

In the similar way, we can prove that $I_2(\Lambda_y^2)\subseteq I_2(\Lambda_x^2)$ (3.18)

From equation (3.17) and equation (3.18), we have $I_2(\Lambda_x^2)=I_2(\Lambda_y^2)$.

Let $\alpha\in I_2(\Gamma_x^2)$. Then $K_3=\{(m,n): N(x_{mn}-\alpha, z)_p<\varepsilon\}\notin I_2$ for each $\varepsilon>0$ and non zero $z\in X$ and $K_4=\{(m,n): m,n\in K_3 and x_{mn}=y_{mn}\}\notin I_2$

$\Rightarrow K_4\subset\{(m,n): N(y_{mn}-\alpha, z)_p<\varepsilon\}$, for each non zero $z\in X$. This shows that, for each $\varepsilon>0$ and non zero $z\in X$, $\{(m,n): N(y_{mn}-\alpha, z)_p<\varepsilon\}\notin I_2$

$\Rightarrow \alpha$ is a I_2-cluster point of $y=(y_{ij})$ in X.

$\Rightarrow \alpha\in I_2(\Gamma_y^2)$ and hence $I_2(\Gamma_x^2)\subseteq I_2(\Gamma_y^2)$ (3.19)

In the similar way, we can easily show that $I_2(\Gamma_y^2)\subseteq I_2(\Gamma_x^2)$ (3.20)

From equation (3.19) and equation (3.20), $I_2(\Gamma_x^2)=I_2(\Gamma_y^2)$.

Corollary 3.21: Let $(X, N(\bullet,\bullet)_p)$ be a $p-$adic linear 2-normed space and M_2^2 be the set of all bounded double sequences of X with norm

$N(x)=\sup_{m,n} N(x_{mn}, z)_p$ for each $z\in X$, where $x=(x_{mn})$ (3.22)

Then M_2^2 is a normed linear space.

Proof: Let $x=(x_{ij})$, $y=(y_{ij})\in M_2^2$ and c_1, c_2 be two scalars in a field K. Then $x=(x_{ij})$ and $y=(y_{ij})$ are bounded double sequences in $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$. By the definition of bounded sequence, for each non zero $z\in X$ and for all $i,j\in N$, there exists $M_1, M_2>0$ such that $N(x_{ij},z)_p<M_1$ and $N(y_{ij},z)_p<M_2$.

Now $N(c_1x_{ij}+c_2y_{ij},z)_p \le N(c_1x_{ij},z)_p+N(c_2y_{ij},z)_p$

$$= |c_1|N(x_{ij},z)_p + |c_2|N(y_{ij},z)_p$$

$$< |c_1|M_1+|c_2|M_2$$

= C, where C=$|c_1|M_1+|c_2|M_2$

Therefore there exists C>0 such that $N(c_1x_{ij}+c_2y_{ij},z)_p<C$, for each non zero $z\in X$ and for all $i,j\in N$.

$\Rightarrow (c_1x_{ij}+c_2y_{ij})$ is also bounded double sequence in X.

$\Rightarrow (c_1x_{ij}+c_2y_{ij})\in M_2^2$, for any two scalars c_1, c_2 in a field K.

Thus M_2^2 is a linear space.Now to complete the proof of the corollary (3.21) it is required show that equation (3.22) is a norm on M_2^2.

(i) $N(x)=0 \Leftrightarrow \sup_{m,n} N(x_{mn},z)_p=0$, for each $z\in X$

$\Leftrightarrow x_{mn}=0$, for each $m,n\in N$

$\Leftrightarrow x=0$

(ii) $N(ax)=\sup_{m,n} N(ax_{mn},z)_p=\sup_{m,n}|a|N(x_{mn},z)_p$

$=|a|\sup_{m,n} N(x_{mn},z)_p$

$=|a|N(x)$, for each $x\in M_2^2$ and $a\in K$

(iii) $N(x+y)=\sup_{m,n} N(x_{mn}+y_{mn},z)_p \le \sup_{m,n} N(x_{mn},z)_p+\sup_{m,n} N(y_{mn},z)_p$

$=N(x)+N(y)$, for $x,y\in M_2^2$

Thus M_2^2 is a normed linear space with respect to the norm equation (3.22).

Theorem 3.23: Let $(X, N(\bullet,\bullet)_p)$ be a $p-$adic 2-Banach space. If I_2 be a non trivial admissible ideal of $N\times N$ and $M_{I_2}^2$ denotes the set of all bounded I_2-convergent double sequences of X. Then the set $M_{I_2}^2$ is a closed linear subspace of the normed linear space M_2^2.

Proof: From Corollary (3.21), $M_{I_2}^2$ is a linear subspace of M_2^2. Now we have to show that $M_{I_2}^2$ is closed in M_2^2.

Let $(x^n)_{n\in N}$ be a Cauchy sequence in $M_{I_2}^2$ such that $(x^n)_{n\in N}$ converges to x and

$x\in M_2^2$. Since $x^n \in M_{I_2}^2$, for each n there exists an element $a_n \in X$ such that $I_2 - \lim_{i,j\to\infty} x_{ij}^n = a_n, n\in N$ and $x^n = (x_{ij})_{i,j\in N}$

To prove this Theorem, it is sufficient to show that $x\in M_{I_2}^2$. For this we need show that

(i) $(a_n)_{n\in N}$ converges to a in X

(ii) $I_2 - \lim_{i,j\to\infty} x_{ij} = a$

Proof of (i): Since $(x^n)_{n\in N}$ is Cauchy sequence in $M_{I_2}^2$, therefore for each $\varepsilon > 0$ and $z\in X$, there exists $N_0 \in N$ such that for each $q \ge r \ge N_0$, we have $N(x^q - x^r, z)_p < \frac{\varepsilon}{3}$.

Now since $x^q, x^r \in M_{I_2}^2$, so $I_2 - \lim x_{ij}^q = a_q$ and $I_2 - \lim x_{ij}^r = a_r$

$\Rightarrow A_q = \{(i,j)\in N\times N : N(x_{ij}^q - a_q, z)_p \ge \frac{\varepsilon}{3}\} \in I_2$ and

$A_r = \{(i,j)\in N\times N : N(x_{ij}^r - a_r, z)_p \ge \frac{\varepsilon}{3}\} \in I_2$, for each non zero $z\in X$.

$\Rightarrow A_q^c = \{(i,j)\in N\times N : N(x_{ij}^q - a_q, z)_p < \frac{\varepsilon}{3}\} \in F(I_2)$ and

$A_r^c = \{(i,j)\in N\times N : N(x_{ij}^r - a_r, z)_p < \frac{\varepsilon}{3}\} \in F(I_2)$, for each $z\in X$.

$\Rightarrow A_q^c \cap A_r^c \in F(I_2)$

Since I_2 is non trivial and admissible so $A_q^c \cap A_r^c$ must be non empty set. Choose $(m_0, n_0) \in A_q^c \cap A_r^c$ and therefore, for each $z\in X$.

$N(x_{m_0n_0}^q - a_q, z)_p < \frac{\varepsilon}{3}$ and $N(x_{m_0n_0}^r - a_r, z)_p < \frac{\varepsilon}{3}$.

$$\begin{aligned}
\text{Now } N(a_q - a_r, z)_p &= N(a_q - x_{m_0n_0}^q + x_{m_0n_0}^q + x_{m_0n_0}^r - x_{m_0n_0}^r - a_r, z)_p \\
&= N((a_q - x_{m_0n_0}^q) + (x_{m_0n_0}^q - x_{m_0n_0}^r) + (x_{m_0n_0}^r - a_r), z)_p \\
&\le N(a_q - x_{m_0n_0}^q, z)_p + N(x_{m_0n_0}^q - x_{m_0n_0}^r, z)_p + N(x_{m_0n_0}^r - a_r, z)_p \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \text{ for each } z\in X \text{ and } q \ge r \ge N_0
\end{aligned}$$

Therefore $(a_n)_{n\in N}$ is a Cauchy sequence in $p-$adic 2-Banach space X and it must converges to an element $a\in X$. Hence $(a_n)_{n\in N}$ converges to $a\in X$ with respect to $N(\bullet,\bullet)_p$.

Proof of (ii): Let $\delta > 0$ .Since $(x^n)_{n\in N}$ converges to x, there exists $q\in N$ such that

$N(x^q - x, z)_p < \frac{\delta}{3}$, for each $z \in X$ (3.24)

The number q can be chosen in such a way that together with equation (3.24) the inequality $N(a_q - a, z)_p < \frac{\delta}{3}$, for each $z \in X$ also holds.

Since $I_2 - \lim_{i,j\to\infty} x_{ij}^q = a_q$, therefore $A_q = \{(i, j) \in N \times N : N(x_{ij}^q - a_q, z)_p \geq \frac{\delta}{3}\} \in I_2$ for each non zero $z \in X$.

$\Rightarrow A_q^c = \{(i, j) \in N \times N : N(x_{ij}^q - a_q, z)_p < \frac{\delta}{3}\} \in F(I_2)$, for each $z \in X$.

For each $(i, j) \in A_q^c$,
$$N(x_{ij} - a, z)_p = N(x_{ij} - x_{ij}^q + x_{ij}^q - a_q + a_q - a, z)_p$$
$$= N((x_{ij} - x_{ij}^q) + (x_{ij}^q - a_q) + (a_q - a), z)_p$$
$$\leq N(x_{ij} - x_{ij}^q, z)_p + N(x_{ij}^q - a_q, z)_p + N(a_q - a, z)_p$$
$$< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta, \text{ for each } z \in X.$$

$\Rightarrow$ The set $\{(i, j) \in N \times N : N(x_{ij} - a, z)_p < \delta\} \in F(I_2)$, for each $z \in X$.

$\Rightarrow$ The set $\{(i, j) \in N \times N : N(x_{ij} - a, z)_p \geq \delta\} \in I_2$, for each non zero $z \in X$.

$\Rightarrow I_2 - \lim_{i,j\to\infty} x_{ij} = a$. This completes the proof of the theorem.

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