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Direct Theorems for Modified Baskakov Stancu Operators in Simultaneous Approximation

Prerna Maheshwari Sharma and Sangeeta Garg

Department of Mathematics, SRM University, NCR Campus; Ghaziabad; India.

Mewar University, Chittorgarh (Rajasthan), India

Abstract

In the present paper, we extend our study for modified Baskakov operators defined by Gupta-Srivastava [5]. We introduce modified Baskakov-Stancu type operators and give the moments in terms of hypergeometric series functions. Further, we establish asymptotic formula and error estimation in simultaneous approximation for these operators.

Key Words: Hypergeometric series functions, Stancu type generalization, Modified Baskakov operators, Simultaneous approximation, Voronovskaja type asymptotic formula, Schwarz inequality.

AMS Subject Classification: 41A25, 41A35.

Introduction

In the year 1985, Sahai-Prasad [8] proposed modified Lupas operators for all functions integrable on $[0,\infty)$

$$Q_{n,m}(x) = (n-1)\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^{\infty} p_{n,v}(t)f(t)dt, \qquad \text{... (1.1)}$$

where

$$p_{n,v}(t) = \binom{n+v-1}{v} t^v (1+t)^{-(n+v)} = \frac{(n)_v}{v!}\frac{x^v}{(1+x)^{n+v}}.$$

Here $(n)_v$ is rising factorial (Pochhammer symbol) and defined as

$$(n)_v = n(n+1)(n+2)\dots(n+v-1) = \frac{(n+v-1)!}{(n-1)!}.$$

In 1991, Sinha et al. [9] studied it as modified Baskakov operators. Several researchers- Aniol et al. [1], Gupta et al. [2], [3], [4] estimated the rate of convergence for these operators. In 2009, some approximation properties of modified Baskakov operators were discussed by P. Maheshwari [6]. Recently Maheshwari-Sharma [7] have obtained approximations for q-Baskakov Beta Stancu type operators. Srivastava et al. [10], [11] also studied some approximation properties of linear positive operators. Motivating the recent studies, we extend our study and find some direct results on modified Baskakov-Stancu operators using hypergeometric series functions. We write the operator (1.1) as

$$Q_n(f,x) = (n-1)\sum_{v=0}^{\infty} \frac{(n)_v}{v!}\frac{x^v}{(1+x)^{n+v}} \int_0^{\infty} \frac{(n)_v}{v!}\frac{t^v}{(1+t)^{n+v}} f(t)dt$$

$$= (n-1)\int_0^{\infty} \frac{f(t)}{[(1+x)(1+t)]^n} \sum_{v=0}^{\infty} \frac{(n)_v (n)_v}{(v!)^2}\left[\frac{xt}{(1+x)(1+t)}\right]^v dt,$$

by applying hypergeometric series

$${}_2F_1(a,b;c;x) = \sum_{v=0}^{\infty} \frac{(a)_v (b)_v}{(c)_v}.\frac{x^v}{v!} \qquad \text{and the identity } (1)_v = 1.$$

We have

$$Q_n(f,x) = (n-1)\int_0^{\infty} \frac{f(t)}{[(1+x)(1+t)]^n}\ {}_2F_1\left(n,n;1;\frac{xt}{(1+x)(1+t)}\right)dt. \quad \dots(1.2)$$

Now in (1.2), applying Pfaff-Kummer's transformation, defined below

$${}_2F_1(a,b;c;x) = (1-x)^{-a}\ {}_2F_1\left(a,c-b;c;\frac{x}{1-x}\right). \quad \dots.(1.3)$$

We have

$$Q_n(f,x) = (n-1)\int_0^{\infty} \frac{f(t)}{(1+x+t)^n}\ {}_2F_1\left(n,1-n;1;\frac{-xt}{1+x+t}\right)dt. \quad \dots(1.4)$$

This is the required alternate form of the operators (1.1) in terms of hypergeometric series functions. In 1983, Stancu type generalization [12] of Bernstein operators based on two parameters α and β satisfying the condition $0 \le \alpha \le \beta$, has been studied by several authors. Motivating the recent work, we propose Stancu type generalization of modified Baskakov operators as

$$Q_n^{\alpha,\beta}(f,x) = (n-1)\int_0^{\infty} f\left(\frac{nt+\alpha}{n+\beta}\right)\ {}_2F_1\left(n,1-n;1;\frac{-xt}{1+x+t}\right)\frac{dt}{(1+x+t)^n}. \quad \dots(1.5)$$

As a special case, if $\alpha = \beta = 0$, modified Baskakov-Stancu operators (1.5) reduce to modified Baskakov operators defined in (1.4). Further we consider

$C_\gamma[0,\infty) = \{f \in C[0,\infty): f(t) = O(t^\gamma), \gamma > 0\}$

so that the operators $Q_n^{\alpha,\beta}$ are well defined.

In the present article, we study direct theorems including Voronovskaja type asymptotic formula and an error estimation in simultaneous approximation theory for MBS operators (1.5).

Moment Estimation and Auxiliary Results

In this section we estimate certain basic results with the use of hypergeometric series functions.

Lemma 1. For $n > 0$ *and* $r > -1$, we have

$$Q_n(t^r,x) = \frac{\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n-1)}(1+x)^r\ {}_2F_1\left(1-n,-r;1;\frac{x}{1+x}\right).$$

Proof. Substituting $f(t) = t^r$ and $t = (1+x)u$ in (1.5),

$$Q_n(t^r,x) = (n-1)\int_0^\infty \frac{u^r}{(1+u)^n}(1+x)^{r-n+1}\ {}_2F_1\left(n,1-n;1;\frac{-ux}{1+u}\right)dt$$

$$= (n-1)\sum_{v=0}^\infty \frac{(n)_v(1-n)_v}{(v!)^2}(-x)^v(1+x)^{r-n+1}\int_0^\infty \frac{u^{r+v}}{(1+u)^{n+v}}du$$

$$= (1+x)^{r-n+1}\sum_{v=0}^\infty \frac{\Gamma(n+v)(1-n)_v}{\Gamma(n-1)(v!)^2}(-x)^v \times \frac{\Gamma(v+r+1)\Gamma(n-r-1)}{\Gamma(n+v)}$$

Using$\Gamma(n+v+1) = \Gamma(n+1)(n+1)_v$, we get

$$Q_n(t^r,x) = (1+x)^{r-n+1}\sum_{v=0}^\infty \frac{(1-n)_v}{\Gamma(n-1)(v!)^2}(-x)^v\Gamma(n-r-1)\Gamma(r+1)(r+1)_v$$

$$= \frac{\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n-1)}(1+x)^{r-n+1}\sum_{v=0}^\infty \frac{(1-n)_v(r+1)_v}{(v!)^2}(-x)^v$$

$$= \frac{(n-r-2)!\,\mathrm{r}!}{(n-2)!}(1+x)^{r-n+1}\ {}_2F_1(1-n,1+r;1;-x).$$

Applying Pfaff-Kummer's transformation, we have

$$Q_n(t^r,x) = \frac{(n-r-2)!\,\mathrm{r}!}{(n-2)!}(1+x)^r\ {}_2F_1\left(1-n,-r;1;\frac{x}{1+x}\right)$$

$$= \frac{\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n-1)}(1+x)^r\ {}_2F_1\left(1-n,-r;1;\frac{x}{1+x}\right).$$

Lemma 2. For $0 \leq \alpha \leq \beta$, we have

$$Q_n^{\alpha,\beta}(f,x) = x^r \frac{n^r}{(n+\beta)^r}\frac{(n+r-1)!\,(n-r-2)!}{(n-1)!\,(n-2)!} + r\frac{n^r}{(n+\beta)^r}[r\frac{(n+r-2)!\,(n-r-2)!}{(n-1)!\,(n-2)!}$$
$$+\frac{\alpha}{n}\frac{(n+r-2)!\,(n-r-1)!}{(n-1)!\,(n-2)!}]x^{r-1} + \frac{n^{r-1}}{(n+\beta)^r}\frac{r(r-1)}{2}\alpha^2 \times$$
$$\left[\frac{(r-1)}{\alpha}\frac{(n+r-3)!\,(n-r-1)!}{(n-1)!\,(n-2)!} + \frac{1}{n}\frac{(n+r-3)!\,(n-r)!}{(n-1)!\,(n-2)!}\right] + O(n^{-2}).$$

Proof. From the relation (1.5)

$$Q_n^{\alpha,\beta}(f,x) = (n-1)\int_0^\infty \left(\frac{nt+\alpha}{n+\beta}\right)^r \ {}_2F_1\left(n,1-n;1;\frac{-xt}{1+x+t}\right)\frac{dt}{(1+x+t)^n}$$
$$= \sum_{j=0}^{r}\binom{r}{j}\frac{n^j\alpha^{r-j}}{(n+\beta)^r}(n-1)\int_0^\infty \frac{t^j}{(1+x+t)^n}\ {}_2F_1\left(n,1-n;1;\frac{-xt}{1+x+t}\right)dt$$
$$= \sum_{j=0}^{r}\binom{r}{j}\frac{n^j\alpha^{r-j}}{(n+\beta)^r}\ Q_n\left(t^j,x\right), \qquad (from (1.2))$$
$$= \frac{1}{(n+\beta)^r}\left[n^r\,Q_n(t^r,x) + rn^{r-1}\alpha\,Q_n(t^{r-1},x) + \frac{r(r-1)}{2}n^{r-2}\alpha^2\,Q_n(t^{r-2},x)\right] + O(n^{-2}).$$

Using Lemma 1, we have

$$Q_n^{\alpha,\beta}(f,x) = \frac{1}{(n+\beta)^r}[n^r\left\{\frac{(n-r-2)!\,(n+r-1)!}{(n-1)!\,(n-2)!}x^r + \frac{(n-r-2)!\,(n+r-1)!}{(n-1)!\,(n-2)!}r^2x^{r-1}\right\}$$
$$+\alpha n^{r-1}\left\{\frac{(n-r-1)!\,(n+r-2)!}{(n-1)!\,(n-2)!}rx^{r-1} + (r-1)^2\frac{(n-r-1)!\,(n+r-3)!}{(n-1)!\,(n-2)!}x^{r-2}\right\}$$
$$+\frac{r(r-1)}{2}n^{r-2}\alpha^2\frac{(n-r)!\,(n+r-3)!}{(n-1)!\,(n-2)!}x^{r-2}] + O(n^{-2}).$$

Collecting the coefficients of x^r, x^{r-1} and x^{r-2}, we obtain the required result.

Lemma 3. [5] *For* $m \in N \cup \{0\}$, if

$$U_{n,m}(x) = \sum_{v=0}^{\infty} p_{n,v}(x)\left(\frac{v}{n}-x\right)^m,$$

then $U_{n,0}(x) = 1$ and $U_{n,1}(x) = 0$ and the recurrence formula

$$nU_{n,m+1}(x) = x(1+x)\left[U'_{n,m}(x) + mU_{n,m-1}(x)\right]$$

Consequently $U_{n,m}(x) = O\left(n^{-[m+1]/2}\right)$, where $[\zeta]$ is integral part of ζ.

Lemma 4. We define the central moments as

$$M_{n,m}^{\alpha,\beta}(f,x)=Q_n^{\alpha,\beta}((t-x)^m,x)=(n-1)\sum_{v=0}^{\infty}p_{n,v}(x)\int_0^{\infty}p_{n,v}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt,$$

then we have $M_{n,0}^{\alpha,\beta}(f,x)=1$ and $M_{n,1}^{\alpha,\beta}(f,x)=\frac{n(1-x+nx)-(n-2)(\alpha-\beta x)}{(n-2)(n+\beta)}$,
for all $n\in N$. Moreover the following recurrence relation holds

$$(n-m-2)\left(\frac{n+\beta}{n}\right)M_{n,m+1}^{\alpha,\beta}(x)$$

$$=x(1+x)\left[M_{n,m}^{\prime,\alpha,\beta}(x)+mM_{n,m-1}^{\alpha,\beta}(x)\right]+\left(\frac{\alpha}{n+\beta}-x\right)\left[\left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta}-x\right)-1\right]M_{n,m-1}^{\alpha,\beta}(x)$$

$$+\left[(m+nx+1)+\left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta}-x\right)(n-2m-2)\right]M_{n,m}^{\alpha,\beta}(x)$$

and $M_{n,m}(x)=O\left(n^{-[m+1]/2}\right)$ for all $x\in[0,\infty)$, where $[\alpha]$ is integral part of α.

Proof. By the definition of the operators (1.5), we get $M_{n,0}^{\alpha,\beta}(x)=1$. Other moments can also be obtained easily. For the recurrence relation, we follow as

$$M_{n,m}^{\prime,\alpha,\beta}(x)+mM_{n,m-1}^{\alpha,\beta}(x)=(n-1)\sum_{v=0}^{\infty}p'_{n,v}(x)\int_0^{\infty}p_{n,v}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt.$$

Using $x(1+x)p'_{n,v}(x)=(v-nx)p_{n,v}(x),$ we get

$$x(1+x)\left[M_{n,m}^{\prime,\alpha,\beta}(x)+mM_{n,m-1}^{\alpha,\beta}(x)\right]$$

$$=(n-1)\sum_{v=0}^{\infty}(v-nx)p_{n,v}(x)\int_0^{\infty}p_{n,v}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt$$

$$=(n-1)\sum_{v=0}^{\infty}(v-nx)p_{n,v}(x)\int_0^{\infty}[(v-nt)+nt-nx]p_{n,v}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt$$

$$=(n-1)\sum_{v=0}^{\infty}p_{n,v}(x)\int_0^{\infty}t(1+t)p_{n,v}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt+n(n-1)$$

$$\times\sum_{v=0}^{\infty}(v-nx)p_{n,v}(x)\int_0^{\infty}tp_{n,v}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt-nxM_{n,m}^{\alpha,\beta}(x).$$

Substituting $t=\frac{n+\beta}{n}\left[\left(\frac{nt+\alpha}{n+\beta}-x\right)-\left(\frac{\alpha}{n+\beta}-x\right)\right]$, we have

$$x(1+x)\left[M_{n,m}^{\prime,\alpha,\beta}(x)+mM_{n,m-1}^{\alpha,\beta}(x)\right]$$

$$=\left(\frac{n+\beta}{n}\right)[(n-1)\sum_{v=0}^{\infty}p_{n,v}(x)\int_0^{\infty}p'_{n,v}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1}dt-\left(\frac{\alpha}{n+\beta}-x\right)(n-1)$$

$$\times \sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} p'_{n,v}(t) \left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt] + \left(\frac{n+\beta}{n}\right)^2 [(n-1) \sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} p'_{n,v}(t)$$

$$\times \left(\frac{nt+\alpha}{n+\beta} - x\right)^{m+2} dt + \left(\frac{\alpha}{n+\beta} - x\right)^2 (n-1) \sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} p'_{n,v}(t) \left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt$$

$$-2\left(\frac{\alpha}{n+\beta} - x\right)(n-1) \sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} p'_{n,v}(t) \left(\frac{nt+\alpha}{n+\beta} - x\right)^{m+1} dt] + (n+\beta)$$

$$\times \left[M_{n,m+1}^{\alpha,\beta}(x) - \left(\frac{\alpha}{n+\beta} - x\right) M_{n,m}^{\alpha,\beta}(x)\right] - nx M_{n,m}^{\alpha,\beta}(x)$$

$$= \left(\frac{n+\beta}{n}\right)\left[-(m+1)M_{n,m}^{\alpha,\beta}(x) + \left(\frac{\alpha}{n+\beta} - x\right) m M_{n,m-1}^{\alpha,\beta}(x)\right] + \left(\frac{n+\beta}{n}\right)^2 [-(m+2)$$

$$\times M_{n,m+1}^{\alpha,\beta}(x) - m\left(\frac{\alpha}{n+\beta} - x\right)^2 M_{n,m-1}^{\alpha,\beta}(x) + 2\left(\frac{\alpha}{n+\beta} - x\right)(m+1) M_{n,m}^{\alpha,\beta}(x)]$$

$$+(n+\beta)\left[M_{n,m+1}^{\alpha,\beta}(x) - \left(\frac{\alpha}{n+\beta} - x\right) M_{n,m}^{\alpha,\beta}(x)\right] - nx M_{n,m}^{\alpha,\beta}(x).$$

On arranging the like terms, we get the desired recurrence formula.

Lemma 5. [5] There exists the polynomial $q_{i,j,r}(x)$ on $[0,\infty)$, independent of n and v such that

$$x^r (1+x)^r \frac{d^r}{dx^r} p_{n,v}(x) = \sum_{\substack{2i+j \le r, \\ i,j \ge 0}} n^i (v - nx)^j \, q_{i,j,r}(x) p_{n,v}(x).$$

Direct Estimates

In this section, we present some important estimation theorems in simultaneous approximation.

Theorem1. Let f $\in C_\gamma[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$, taking the derivative of order $(r+2)$ at fixed $x \in [0,\infty)$. Consider $f(t) = O(t^\gamma)$ as $t \to \infty$, for some $\gamma > 0$, we have

$$\lim_{n\to\infty} n\left[Q_{n,\alpha,\beta}^{(r)}(f,x) - f^{(r)}(x)\right] = r(1+r-\beta) f^{(r)}(x) + [(1+r+\alpha) + x(2+2r-\beta)]$$

$$\times f^{(r+1)}(x) + x(1+x) f^{(r+2)}(x).$$

Proof. Taylor expansion of f is given as

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \epsilon(t,x)(t-x)^{r+2},$$

where$\epsilon(t,x)\to 0$ and $\epsilon(t,x)\to O\big((t-x)^{\delta}\big)$ as $t\to\infty$ for some $\delta>0$. Hence

$$n\Big[Q_{n,\alpha,\beta}^{(r)}(f,x)-f^{(r)}(x)\Big]$$
$$=n\left[\left\{\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}Q_{n,\alpha,\beta}^{(r)}\big((t-x)^{i},x\big)-f^{(r)}(x)\right\}+Q_{n,\alpha,\beta}^{(r)}(\epsilon(t,x)(t-x)^{r+2},x)\right]$$
$$=J_1+J_2. \qquad \dots(3.1)$$

From Lemma 2, we get

$$J_1=n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=r}^{i}\binom{i}{j}(-x)^{i-j}Q_{n,\alpha,\beta}^{(r)}\big(t^{j},x\big)-nf^{(r)}(x)$$
$$=n\frac{f^{(r)}(x)}{r!}\Big[Q_{n,\alpha,\beta}^{(r)}(t^{r},x)-r!\Big]+n\frac{f^{(r+1)}(x)}{(r+1)!}[(r+1)(-x)Q_{n,\alpha,\beta}^{(r)}(t^{r},x)+Q_{n,\alpha,\beta}^{(r)}(t^{r+1},x)]$$
$$+n\frac{f^{(r+2)}(x)}{(r+2)!}\left[\frac{(r+1)(r+2)}{2!}x^{2}Q_{n,\alpha,\beta}^{(r)}(t^{r},x)-(r+2)xQ_{n,\alpha,\beta}^{(r)}(t^{r+1},x)+Q_{n,\alpha,\beta}^{(r)}(t^{r+2},x)\right]$$
$$=nf^{(r)}(x)\left[\frac{n^{r}}{(n+\beta)^{r}}\frac{(n+r-1)!\,(n-r-2)!}{(n-1)!\,(n-2)!}-1\right]+\frac{nf^{(r+1)}(x)}{(r+1)!}\times\frac{n^{r}}{(n+\beta)^{r}}[(r+1)(-x)$$
$$\times\frac{(n+r-1)!\,(n-r-2)!}{(n-1)!\,(n-2)!}r!+\frac{n}{n+\beta}(\frac{(n+r)!\,(n-r-3)!}{(n-1)!\,(n-2)!}(r+1)!\,x+(r+1)^{2}$$
$$\times\frac{(n+r-1)!\,(n-r-3)!}{(n-1)!\,(n-2)!}r!)+\frac{(r+1)\alpha}{n+\beta}\frac{(n+r-1)!\,(n-r-2)!}{(n-1)!\,(n-2)!}r!]+\frac{nf^{(r+2)}(x)}{(r+2)!}$$
$$\times\frac{n^{r}}{(n+\beta)^{r}}[\frac{(r+1)(r+2)}{2!}\frac{(n+r-1)!\,(n-r-2)!}{(n-1)!\,(n-2)!}r!\,x^{2}-(r+2)x\{\frac{n}{n+\beta}$$
$$\times\left(\frac{(n+r)!\,(n-r-3)!}{(n-1)!\,(n-2)!}(r+1)!\,x+\frac{(n+r-1)!\,(n-r-3)!}{(n-1)!\,(n-2)!}r!\right)+\frac{(r+1)\alpha}{n+\beta}$$
$$\times\frac{(n+r-1)!\,(n-r-2)!}{(n-1)!\,(n-2)!}r!\}+\left(\frac{n}{n+\beta}\right)^{2}\{\frac{(n+r+1)!\,(n-r-4)!}{(n-1)!\,(n-2)!}\frac{(r+2)!}{2}x^{2}+(r+2)^{2}$$
$$\times\frac{(n+r)!\,(n-r-2)!}{(n-1)!\,(n-2)!}(r+1)!\,x\}+(r+2)\alpha\frac{n}{(n+\beta)^{2}}\{\frac{(n+r)!\,(n-r-3)!}{(n-1)!\,(n-2)!}(r+1)!\,x+$$
$$(r+1)^{2}\frac{(n+r-1)!\,(n-r-3)!}{(n-1)!\,(n-2)!}r!\}+\frac{(r+1)(r+2)}{2!}\frac{\alpha^{2}}{(n+\beta)^{2}}\frac{(n+r-1)!\,(n-r-2)!}{(n-1)!\,(n-2)!}r!]$$
$$+O(n^{-2}).$$

Substituting $r=1,2,\ldots$ in the coefficients of $f^{(r)}$ and then taking limit as $n\to\infty$, we get the coefficient of $f^{(1)}(x)=(2-\beta)=1(1+1-\beta)$,

the coefficient of $f^{(2)}(x)=(6-2\beta)=2(1+2-\beta)$ and so on.

Thus by induction, the coefficients of $f^{(r)}(x)$, $f^{(r+1)}(x)$ and $f^{(r+2)}(x)$ are $r(1+r-\beta)$, $(1+r+\alpha)+x(2+2r-\beta)$ and $x(1+x)$ respectively. In order to complete the proof, we have to show that $J_2\to 0$ as $n\to\infty$. Therefore using Lemma 5,

$$|J_2| \le n^{i+1}(n-1) \sum_{\substack{2i+j\le r,\\ i,j\ge 0}} \frac{|q_{i,j,r}(x)|}{x^r(1+x)^r} \sum_{v=0}^{\infty} p_{n,v}(x)\,|v-nx|^j$$

$$\times \sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} p_{n,v}(t)|\epsilon(t,x)| \left|\frac{nt+\alpha}{n+\beta}-x\right|^{r+2} dt.$$

Since $\epsilon(t,x) \to 0$ as $t \to x$ for a given $\epsilon > 0$, there exists $\delta > 0$ such that $|\epsilon(t,x)| < \epsilon$ whenever $|t-x| < \delta$. Further, if $\lambda > max\{\gamma, r+2\}$, where λ is any integer. Then for all $|t-x| \ge \delta$, we can find a constant $M > 0$ such that

$$|\epsilon(t,x)| \left|\frac{nt+\alpha}{n+\beta}-x\right|^{r+2} \le M \left|\frac{nt+\alpha}{n+\beta}-x\right|^{\lambda}.$$

Therefore from above, we have

$$|J_2| \le C_1 \sum_{\substack{2i+j\le r,\\ i,j\ge 0}} n^{i+1}(n-1) \sum_{v=0}^{\infty} p_{n,v}(x)\,|v-nx|^j \times$$

$$\left\{ \int_{|t-x|<\delta} \epsilon p_{n,v}(t) \left|\frac{nt+\alpha}{n+\beta}-x\right|^{r+2} dt + \int_{|t-x|\ge\delta} M p_{n,v}(t) \left|\frac{nt+\alpha}{n+\beta}-x\right|^{r+2} dt \right\}$$

$$= J_3 + J_4.$$

Applying Schwarz inequality for integration and summation, we get

$$|J_3| \le \epsilon C_1 \sum_{\substack{2i+j\le r,\\ i,j\ge 0}} n^{i+1} \sum_{v=0}^{\infty} p_{n,v}(x)\,|v-nx|^j \left\{(n-1)\int_0^{\infty} p_{n,v}(t)dt\right\}^{1/2}$$

$$\times \left\{(n-1)\int_0^{\infty} p_{n,v}(t) \left|\frac{nt+\alpha}{n+\beta}-x\right|^{2r+4} dt\right\}^{1/2}$$

$$\le \epsilon C_1 \sum_{\substack{2i+j\le r,\\ i,j\ge 0}} n^{i+1} \left\{\sum_{v=0}^{\infty}(n-1)p_{n,v}(x)\,|v-nx|^{2j}\right\}^{1/2}$$

$$\times \left\{(n-1)\sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} p_{n,v}(t) \left|\frac{nt+\alpha}{n+\beta}-x\right|^{2r+4} dt\right\}^{1/2}.$$

For arbitrary ϵ, by using Lemma 3 and Lemma 4, we get

$$\|J_3\| \le \epsilon C_1 \sum_{\substack{2i+j\le r,\\ i,j\ge 0}} n^{i+1}\, O\left(n^{j/2}\right) O\left(n^{-[r+2]/2}\right) = \epsilon O(1) = o(1).$$

Applying Schwarz inequality in J_4,

$$|J_4| \le C_2 \sum_{\substack{2i+j\le r,\\ i,j\ge 0}} n^{i+1}(n-1)\sum_{v=0}^{\infty} p_{n,v}(x)\,|v-nx|^j \int_0^{\infty} p_{n,v}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^{\lambda} dt$$

$$\le C_2 \sum_{\substack{2i+j\le r,\\ i,j\ge 0}} n^{i+1}\left\{(n-1)\sum_{v=0}^{\infty} p_{n,v}(x)\,|v-nx|^{2j}\right\}^{1/3}$$

$$\times\left\{(n-1)\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^{\infty} p_{n,v}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^{2\lambda} dt\right\}^{1/2}.$$

.Using Lemma 3 and Lemma 4, we obtain

$$\|J_4\| \le C_2 \sum_{\substack{2i+j\le r,\\ i,j\ge 0}} n^{i+1}\, O\left(n^{j/2}\right)O\left(n^{-\lambda/2}\right) = C_2 O\left(n^{(r+2-\lambda)/2}\right) = o(1),$$

where C_1 and C_2 are arbitrary constants. Now $J_2 \to 0$ as $n \to \infty$. Again combining the estimates J_1 and J_2, we get the desired result. Hence the theorem is completed.

Theorem 2. Let f $\in C_{\gamma}[0,\infty)$ for some $\gamma > 0$ and $r \le m \le r+2$. If $f^{(m)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then for n sufficiently large, we have

$$\left\|Q_{n,\alpha,\beta}^{(r)}(f,x) - f^{(r)}(x)\right\|_{C[a,b]} \le K_1 n^{-1}\sum_{i=r}^{m}\left\|f^{(i)}(x)\right\|_{C[a,b]} + K_2 n^{-1/2}\omega\left(f^{(m)}, n^{-1/2}\right) + O(n^{-2})$$

where K_1 and K_2 are constants independent of f and n, $\omega(f,\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$ and $\|.\|_{C[a;b]}$ denotes the sup-norm on $[a,b]$.

Proof. Taylor expansion of f is given by

$$f(t) = \sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}(t-x)^m\chi(t) + h(t,x)\left(1-\chi(t)\right),$$

whereξ is the characteristic function on $(a-\eta, b+\eta)$ and lies between t and x. .Therefore

$$\left\{Q_{n,\alpha,\beta}^{(r)}(f,x) - f^{(r)}(x)\right\}$$

$$= \sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}Q_{n,\alpha,\beta}^{(r)}\left((t-x)^i, x\right) + Q_{n,\alpha,\beta}^{(r)}\left(\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}(t-x)^m\chi(t), x\right)$$

$$+ Q_{n,\alpha,\beta}^{(r)}\left(h(t,x)\left(1-\chi(t)\right), x\right) = E_1 + E_2 + E_3.$$

Applying Lemma 2,

$$E_1=\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}(-x)^{i-j}\frac{d^r}{dx^r}[j\frac{n^j}{(n+\beta)^j}\{j\frac{(n+j-1)!\,(n-j-2)!}{(n-1)!\,(n-2)!}+\frac{\alpha}{n}$$
$$\times\frac{(n+j-2)!\,(n-j-1)!}{(n-1)!\,(n-2)!}\}x^{j-1}+\frac{j(j-1)^2}{2}\alpha\frac{n^{j-1}}{(n+\beta)^j}\{2\frac{(n+j-3)!\,(n-j-1)!}{(n-1)!\,(n-2)!}$$
$$+\frac{\alpha}{n(j-1)}\frac{(n+j-3)!\,(n-j)!}{(n-1)!\,(n-2)!}\}x^{J-2}+O(n^{-2})]-f^{(r)}(x).$$

Consequently, we have

$$\|E_1\|_{C[a,b]}\le K_1n^{-1}\sum_{i=r}^{m}\left\|f^{(i)}(x)\right\|_{C[a,b]}+O(n^{-2})$$

uniformly on $[a,b]$. To estimate E_2, we take

$$|E_2|\le(n-1)\sum_{v=0}^{\infty}\left|p_{n,v}^{(r)}(t)\right|\int_0^{\infty}p_{n,v}(t)\left|\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}\right|\left|\frac{nt+\alpha}{n+\beta}-x\right|^m\chi(t)dt$$

By the continuity of *f*, we have $\left|f^{(m)}(\xi)-f^{(m)}(x)\right|\le\omega\left(f^{(m)},\delta\right)$. Thus

$$|E_2|\le\frac{\omega\left(f^{(m)},\delta\right)}{\mathrm{m}!}(n-1)\sum_{v=0}^{\infty}\left|p_{n,v}^{(r)}(t)\right|\int_0^{\infty}p_{n,v}(t)\left|1-\frac{\frac{nt+\alpha}{n+\beta}-x}{\delta}\right|\left|\frac{nt+\alpha}{n+\beta}-x\right|^m dt$$

$$\le\frac{\omega\left(f^{(m)},\delta\right)}{\mathrm{m}!}\{(n-1)\sum_{v=0}^{\infty}\left|p_{n,v}^{(r)}(t)\right|\int_0^{\infty}p_{n,v}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^m dt$$

$$+(n-1)\sum_{v=0}^{\infty}\left|p_{n,v}^{(r)}(t)\right|\int_0^{\infty}p_{n,v}(t)\delta^{-1}\left|\frac{nt+\alpha}{n+\beta}-x\right|^{m+1}dt.$$

By using Schwarz inequality,

$$(n-1)\sum_{v=0}^{\infty}\left|p_{n,v}^{(r)}(t)\right||v-nx|^j\int_0^{\infty}p_{n,v}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^m dt$$

$$\le\left\{\sum_{v=0}^{\infty}\left|p_{n,v}^{(r)}(t)\right||v-nx|^{2j}\right\}^{\frac{1}{2}}\left\{(n-1)\int_0^{\infty}p_{n,v}(t)dt\right\}^{\frac{1}{2}}\left\{(n-1)\int_0^{\infty}p_{n,v}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^{2m}dt\right\}^{1/2}$$

$$\le\mathrm{O}\left(\mathrm{n}^{\mathrm{j}/2}\right).\mathrm{O}\left(\mathrm{n}^{-\mathrm{m}/2}\right)=\mathrm{O}\left(\mathrm{n}^{(\mathrm{j}-\mathrm{m})/2}\right).$$

Therefore, using Lemma 5

$$(n-1)\sum_{v=0}^{\infty}\left|p_{n,v}^{(r)}(t)\right|\int_0^{\infty}p_{n,v}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^m dt$$

$$\leq \sum_{v=0}^{\infty} \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} n^i |v-nx|^j \frac{|q_{i,j,r}(x)|}{x^r(1+x)^r} (n-1) p_{n,v}(x) \int_0^{\infty} p_{n,v}(t) \left|\frac{nt+\alpha}{n+\beta} - x\right|^m dt$$

$$\leq \sup_{\substack{2i+j\geq r,\\ i,j\geq 0}} \frac{|q_{i,j,r}(x)|}{x^r(1+x)^r} \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} n^i (n-1) \sum_{v=0}^{\infty} p_{n,v}(x)\, |v-nx|^j \int_0^{\infty} p_{n,v}(t) \left|\frac{nt+\alpha}{n+\beta} - x\right|^m dt$$

$$= C \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} n^i\, O\left(n^{(j-m)/2}\right) = O\left(n^{(r-m)/2}\right)$$

uniformly on $[a,b]$, where

$$C = \sup_{\substack{2i+j\geq r,\\ i,j\geq 0}} \sup_{x\in[a,b]} \frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}$$

Taking $\delta = n^{-1/2}$ and using (3.1), we get

$$\|E_2\|_{C[a,b]} \leq \frac{\omega\left(f^{(m)}, n^{-1/2}\right)}{m!} \left[O\left(n^{(r-m)/2}\right) + n^{-1/2} O\left(n^{(r-m-1)/2}\right) + O(n^{-m})\right]$$

$$\leq K_2 O\left(n^{-(r-m)/2}\right) \omega\left(f^{(m)}, n^{-1/2}\right).$$

To estimate E_3, using Lemma 5, we have

$$|E_3| \leq C \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} n^i \sum_{v=0}^{\infty} p_{n,v}(x)\, |v-nx|^j \int_0^{\infty} p_{n,v}(t) |h(t,x)| dt.$$

Since $t \in [0,\infty)\backslash(a-\eta, b+\eta)$, we can choose δ such that $|t-x| \geq \delta$, for all $[a,b]$. Now choosing a constant K such that $|h(t,x)| \leq K\left|\frac{nt+\alpha}{n+\beta} - x\right|^{\mu}$, where $\mu \geq max\{r,m\}$ is an integer. Hence using (3.1), we get $\|E_3\| = O(n^{-s})$ for all $s > 0$ and uniformly on $[a,b]$.

Combining estimates E_1, E_2 and $E_{3,}$ we get the required result.

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