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On semiprime, Prime and Strongly Prime *-bi-ideals in Semigroup with Involution

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Abstract

In [1] Ahsan and Liu has characterized the prime (semiprime) ideals and right ideals for a semigroup. Furthermore G. Szasz[5] has considered semigroups in which ideals are prime. He has also shown that the ideals of a semigroup P are prime if and only if P is intra regular. In this paper we have introduce the concept of semiprime, prime and strongly prime *-bi-ideals in semigroups with involution. Also we have given characterizations for involution semigroups whose bi-ideals are having all the above properties. Furthermore we have investigated their structural properties by attaching involution on their corresponding ground semigroups. For every involution semigroup S we denote *-bi-ideal for bi-ideals in involution semigroup.

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1. Introduction

In [1] Ahsan and Liu has characterized the prime (semiprime) ideals and right ideals for a semigroup. Furthermore G. Szasz [5] has considered semigroups in which ideals are prime. All definitions and fundamental concepts concerning semigroups and their involution properties can be found in [2] and [4]. In this section we have given related definitions based on prime, semiprime and strongly prime *-bi-ideal in involution semigroups. Also we have given some examples. Throughout this paper we have assumed S with a zero element. For different notations and terminologies the reader is referred to [2]. For every involution semigroup S we denote *-bi-ideal for bi-ideals in semigroup. Let B be a $*$-bi-ideal then $B(a)$ denote $*$-bi-ideal generated by an element a.

Definition 1.1: [6] A $*$-semigroup is a set S equipped with a binary operation $.$ and a unary operation $*: S \to S$ satisfying the following three axioms:

(i) $(x.y).z = x.(y.z)$

(ii) $(a^*)^* = a^{**} = a$

(iii) $(ab)^* = b^*a^* \ \forall\, a, b \in S$.

Such a unary operation $*$ is sometimes called an involution, and (S, $\cdot$, $*$) is sometimes called an involution semigroup.

Let A be a non-empty subset of a $*$-semigroup $(S, \cdot, *)$ then A is said to be a $*$-subsemigroup of S if $ab \in A$ for all $a, b \in A$. Where $*$ is taken as inverse of an element. Somewhere $*$ is taken as transpose in matrix sense and somewhere it is taken as inverse in general set theoretical sense. A $*$-subsemigroup B of an involution semigroup S is called a $*$-bi-ideal of S if $BSB \subseteq$B with the condition $B^* \subseteq B$ and $B^* = \{b^* \in S: b \in B\}$[3]. It is well known that intersection of any two $*$-bi-ideal of an involution semigroup S is a $*$-bi-ideal and in more general way intersection of a finite number of $*$-bi-ideal is again a $*$-bi-ideal. Also product of any two $*$-bi-ideal is a $*$-bi-ideal of S.

Definition1.2: Let B be a $*$-bi-ideal of S then B is said to be a prime $*$-bi-ideal of S if $B_1B_2 \subseteq B$ implies that $B_1 \subseteq B$ *or* $B_2 \subseteq B$, where B_1, B_2 both are $*$-bi-ideal of S. Equivalently, $S - B$ is a $*$-semigroup. Analogously we can say that a $*$-bi-ideal B is completely prime if for any two $b_1, b_2 \in B$ implies that $b_1 \in B$ or $b_2 \in B$ where b_1, b_2 being the elements of S. A $*$-bi-ideal B which is completely prime is prime.

Example1.3: An involution semigroup S itself is always a prime $*$-bi-ideal of S.

Examples1.4: Suppose that $S = [0,1[$ be an involution semigroup under the $*$ operation taken as inverse of numbers between 0 and 1with respect to multiplication (Excluding 1) where $a \in [0,1[$. Considering an element $\frac{1}{2}$ with any other subset C_1 such that,

$S = C_1 \cup \{\frac{1}{2}\}$. Now suppose o denote commutative multiplication in such a way that:

$aob = \{ab \text{ ,if } a \in C_1,\ b \in C_1\}$

$aob = \{0 \text{ ,if } a \in C_1,\ \ b = \frac{1}{2}\}$

$aob = \{\frac{1}{4} \text{ ,if } a = \frac{1}{2},\ \ b = \frac{1}{2}\}$

Then the set $B = \{0, \frac{1}{2}\}$ is a prime $*$-bi-ideal of S.

Example 1.5: Suppose that $S = M_{2x2}(\mathbb{Z}_4)$ be the $2x2$ matrix $*$-semigroup over the semigroup $\mathbb{Z}_4$. Then $P(B) = \left\{\begin{pmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{pmatrix} \middle| a_{ij} \in \mathbb{Z}_2\right\}$.

Definition 1.5: Let B be a $*$-bi-ideal of S then B is said to be a semiprime $*$-bi-ideal of S if $\forall B_1 \in B$, $B_1^2 \subseteq B \Rightarrow B_1 \subseteq B$.

Definition 1.6: Let B be a $*$-bi-ideal then B is said to be a strongly prime $*$-bi-ideal of S if for any two $*$-bi-ideals B_1 and B_2 of S we have,

$B_1B_2 \cap B_2B_1 \subseteq B \Rightarrow B_1 \subseteq B$ or $B_2 \subseteq B$.

Remarks: Every strongly prime $*$-bi-ideal of an involution semigroup S is a prime $*$-bi-ideal and every prime $*$-bi-ideal is a semiprime $*$-bi-ideal. A prime $*$-bi-ideal is not necessarily strongly prime and a semiprime $*$-bi-ideal is not necessarily prime.

Lemma 1.7: Now suppose that $StP\ (B)$ denote strongly prime $*$-bi-ideal, $SeP\ (B)$ denote semiprime $*$-bi-ideal and $P(B)$ denote prime $*$-bi-ideal of S. Then $StP(B) \subseteq P(B) \subseteq SeP\ (B)$.

Proof: Suppose that B is a strongly prime $*$-bi-ideal i.e. suppose $B \in$ StP (B). Our aim is to show that $B \in P(B)$. To that end let $CD \subseteq B$. Since B is strongly prime so $C \subseteq B$ or $D \subseteq B$. So B is prime. Which implies that $StP(B) \subseteq P(B)$. Now suppose that $C^2 \subseteq B$. Then clearly, $CC \subseteq B$ and so $C \subseteq B$. Hence B is semiprime. Finally, $P(B) \subseteq SeP\ (B)$. It follows from the definition itself.

Example 1.8: Consider the semigroup $S = \{0,1,2\}$ defined in tabular form as follows:

.	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

Here $*$-bi-ideals of S are $\{0\}$, $\{0,1\}$, $\{0,2\}$ and $\{0,1,2\}$.

Therefore, all $*$-bi-ideals are prime $*$-bi-ideals and hence semiprime $*$-bi-ideals. However, the prime $*$-bi-ideals $\{0\}$ is not strongly $\{0\}$ prime $*$-bi-ideals because

$\{0,1\}\{0,2\} \cap \{0,2\}\{0,1\} = \{0,1\} \cap \{0,2\} = \{0\} \subseteq \{0\}$.

Example 1.9: All $*$-maximal ideals of involution semigroup S are strongly prime $*$-bi-ideals.

Lemma1.10: Let B_1 and B_2 be any two prime $*$-bi-ideals of an involution semigroup S then $B_1 \cap B_2$ is a prime $*$-bi-ideals then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

Proof: It is clear that $B_1B_2 \subseteq B_1 \cap B_2$. Because $B_1 \cap B_2$ is prime $*$-bi-ideals (By assumption) of S so either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$ and hence either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

Theorem 1.11: For an involution semigroup S the following assertions are true:

(i) $B^2 = B$ for every $*$-bi-ideal B of S.

(ii) $B_1 \cap B_2 = B_1B_2 \cap B_2B_1$ for all $*$-bi-ideal B_1 and B_2 of S.

(iii) Each $*$-bi-ideal of S is semiprime.

Proof: $(i) \Rightarrow (ii)$ Let B_1 and B_2 be any two $*$-bi-ideals of S. Then by our hypothesis,

$$B_1 \cap B_2 = (B_1 \cap B_2)^2$$
$$= (B_1 \cap B_2)\ (B_1 \cap B_2)$$
$$\subseteq B_1$$

In the similar way,

We have $B_1 \cap B_2 \subseteq B_2B_1$

$$B_1 \cap B_2 \subseteq B_1B_2 \cap B_2B_1$$

Here it is clear that B_1B_2 and B_2B_1 both are $*$-bi-ideals being the product of $*$-bi-ideals. Moreover $B_1B_2 \cap B_2B_1$ is also a $*$-bi-ideals. And hence,

$$B_1B_2 \cap B_2B_1 = (B_1B_2 \cap B_2B_1)(B_1B_2 \cap B_2B_1)$$
$$\subseteq B_1B_2B_2B_1$$
$$\subseteq B_1SB_1$$
$$\subseteq B_1$$

In the same way we can have,

$$B_1B_2 \cap B_2B_1 \subseteq B_2$$

And hence we can say that,

$$B_1B_2 \cap B_2B_1 \subseteq B_1 \cap B_2$$

Therefore by above two conditions we have,

$$B_1B_2 \cap B_2B_1 = B_1 \cap B_2$$

$(iii) \Rightarrow (iv)$. For this we let that $B_1\ and\ B_2$ be any two $*$-bi-ideals of S such that $B_1^2 \subseteq B$. Which implies by our supposition that,

$$B_1 = B_1 \cap B_1 = B_1B_1 \cap B_1B_1 = B_1^2$$

So,

$B_1 \subseteq B$.

Hence every $*$-bi-ideal is semiprime.

Lemma 1.12: The intersection of family of prime $*$-bi-ideals of a commutative involution semigroup is a semiprime $*$-bi-ideals.

Proof: We'll show this result for two prime $*$-bi-ideals and then generalize for family of prime $*$-bi-ideals. Suppose that B_1 and B_2 be any two prime $*$-bi-ideals of an involution semigroup S. Now for any $x \in S$, $x^2 \in B_1 \cap B_2 \Rightarrow x^2 \in B_1$ and $x^2 \in B_2$. Since B_1 is a prime $*$-bi-ideal of S. Therefore, $x^2 = x.x \in B_1 \Rightarrow x \in B_1$. In the same way, $x^2 = x.x \in B_2 \Rightarrow x \in B_2$. Hence $x \in B_1 \cap B_2$.

Theorem1.13: Let S be a semigroup with involution then the following assertion are true:

(i) $(A)^2 = A$ for every $*$-bi-ideal A of S.

(ii) $A \cap C = AC$ for every $*$-bi-ideals A *and* C of S.

(iii)$B(a) \cap B(b) = B(a)B(b)$ for all $a, b \in S$.

(iv)$B(a) = (B(a))^2$ for all $a \in S$.

(v) $a \in (BaBaB)$ for all $a \in S$.

Proof: (i) is followed by (ii): Consider A and C be $*$-bi-ideal of S. Then $AC \subseteq AS \subseteq A$ in other way $AC \subseteq SC \subseteq C$ i.e; $AC \subseteq A \cap C$. As $A \cap C$ is a $*$-bi-ideal of S[give reference of intersection of bi-ideal in again a bi-ideal] therefore we have,

$A \cap C = (A \cap C)^2 = (A \cap C)(A \cap C) \subseteq AC$.

(ii) is followed by (iii): For this we consider that $a, b \in S$. Since $B(a), B(b)$ are $*$-bi-ideal of S. Hence,

$B(a) \cap B(b) = B(a)B(b)$ by (ii).

(iii) is followed by (iv): Straight forward.

$(iv)$is followed by (v): Let $a \in S$. Here $B(a) = (B(a))^2$ thus we have,

$(B(a))^2 = (B(a))^2 B(a) \subseteq (B(a))^3 \subseteq (B(a))^3 B(a) \subseteq (B(a))^4$.

$\Rightarrow (B(a))^4 \subseteq (B(a))^3$.

Therefore we have,

$$
\begin{aligned}
B(a) = (B(a))^2 &\subseteq (B(a))^3 \subseteq (B(a))^4 \\
&= (B(a))^4 \subseteq (B(a))^3 = (B(a))^3 \\
&\subseteq (SB(a)) \subseteq (B(a)) = (B(a)) \\
&\Rightarrow B(a) = (B(a))^3.
\end{aligned}
$$

In the similar way we can have,

$$
\begin{aligned}
(B(a))^3 &= ((a \cup Sa \cup aS \cup SaS))^3 \\
&\subseteq ((a \cup Sa \cup aS \cup SaS))^2 (a \cup Sa \cup aS \cup SaS) \\
&\subseteq (Sa \cup SaS)(a \cup Sa \cup aS \cup SaS)
\end{aligned}
$$

$$\subseteq (Sa \cup SaS)(a \cup Sa \cup aS \cup SaS) \subseteq SaS$$

$$\Rightarrow (B(a))^4 \subseteq (SaS)(a \cup Sa \cup aS \cup SaS)$$

$$\subseteq (SaSa \cup SaS^2a \cup SaSaS \cup SaS^2aS)$$

$$= (SaSa \cup SaSaS)$$

$$\Rightarrow (B(a))^2 \subseteq (SaSa \cup SaSaS)(a \cup Sa \cup aS \cup SaS)$$

$$\subseteq (SaSaS).$$

And hence finally we have,

$$a \in B(a) = (B(a))^5 \subseteq (SaSaS) = SaSaS.$$

(v) is followed by (i): For this we suppose that B_1be $*$-bi-ideal of S. Next suppose that $x \in B_1^2$. Since B_1 is a $*$-bi-ideal of S. Therefore, $ab \in B_1$.

Further let $x \in B_1$then by (v), $x \in (BxBxB)$. Also since,

$(tx)n \in (SB_1)S \subseteq B_1S \subseteq B_1$, also $xk \in B_1S \subseteq B_1$. Therefore we have, $x \in B_1^2$ for some $t, n, k \in S$.

Theorem 1.14: Let S denote semigroup with involution and B be $*$-bi-ideal of S then B is strongly prime $*$-bi-ideal of S if and only if one of the five equivalent conditions of lemma 1.9 holds in S.

References

[1] J. Ahsan and Zhongkui Liu: Prime and semiprime acts over monoids with zero, *Math. J. Ibaraki Univ.* 33, 9-15 (2001)

[2] J. M. Howie, Fundamentals of semigroup theory, Oxford University Press, London, 1995.

[3] K.I. Beidar, L.MÁRKI, R.MLITZ and WIEGANDT, *Primitive inv olution rings*, Acta Math. Hungar. 109(4) (2005), 357-368

[4] Max-Albert Knus, Alexander Merkurjev, Markus Rost and Jean-Pierre Tigno, AMS Colloquium Publications, Vol. 44, 1998.

[5] Szasz, G., Eine Charakteristic der Primeidealha-lbgroupen, Publ. Math Debrecen 17 (1970), 209-213)

[6] T.E. Nordahl and H.E. Scheiblich, *Regular $*$-semigroups*, semigroup forum vol. 16(1978) 369-377