Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 20, Number 2 (2024), pp. 377-391

http://www.ripublication.com

Fixed Point Theorems for Generalized $\alpha - \phi$ Contractive Mappings of Type A and Type B in Metric-Like Spaces

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Abstract

In this paper, we shall prove some fixed point results for generalized $\alpha - \phi$ contractive mappings of type A and type B in metric-like space. Some examples are also provided to prove the validity our results. In the end, as an application of second order differential equation is also solved by making use of our results.

Keywords: $\alpha -$ admissible mappings, metric-like space, fixed point.

AMS(2010): 47H10, 54H25.

1. Introduction and Preliminaries

The notion of metric-like (dislocated) spaces were introduced by Hitzler and Seda [9] as a generalization of a metric space in 2000. They generalized Banach Contraction Principle [5] in such spaces. Amini-Harandi [4] rediscovered metric-like spaces who also established some fixed point results. Many fixed point results have been proved in metric-like (quasi) spaces, see for example ([1], [2], [11], [12], [14], [16], [17]).

In the following results, $\mathbb{N}^*, \mathbb{R}$ and $\mathbb{R}_0^+$ will denote the set of positive integer numbers, the set of real numbers and the set of non-negative real numbers respectively.

Definition 1.1 [4] Let Y be a non-empty set. A function $\sigma : Y \times Y \to \mathbb{R}_0^+$ is said to be metric-like (dislocated) on Y if for any $u, v, w \in Y$, the following three conditions hold:

$(\sigma 1)$ $\sigma(u, v) = 0 \Longrightarrow u = v$;

$(\sigma 2)$ $\sigma(u, v) = \sigma(v, u)$;

$(\sigma 3)$ $\sigma(u, w) \leq \sigma(u, v) + \sigma(v, w)$.

Then, the pair (Y,σ) is called a metric-like (dislocated) space.

Example 1.2 A trivial example of a metric-like space is the pair $(\mathbb{R}_0^+,\sigma)$, where
$\sigma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is defined by
$\sigma(u,v) = \max\{u,v\}$.
Here, σ is also a partial metric [13].

Example 1.3 Take $Y = \{2,3,4\}$ and consider the metric-like $\sigma : Y \times Y \to \mathbb{R}_0^+$ given by
$\sigma(2,2) = 0, \sigma(3,3) = 1, \sigma(4,4) = \frac{2}{4}, \sigma(2,3) = \sigma(3,2) = \frac{8}{10}$,
$\sigma(3,4) = \sigma(4,3) = \frac{7}{10}, \sigma(2,4) = \sigma(4,2) = \frac{6}{10}$.
Since $\sigma(3,3) \neq 0$,so σ is not a metric and since $\sigma(3,3) > \sigma(2,3)$, so σ is not a partial metric.
Each metric-like σ on Y generates a topology τ_σ on Y whose base is the family of open σ-balls, $B_\sigma(u,\varepsilon)$, where
$B_\sigma(u,\varepsilon) = \{v \in Y: |\sigma(u,v) - \sigma(u,u)| < \varepsilon\}$,
for all $u \in Y$ and $\varepsilon > 0$.

Definition 1.4 [4] Let (Y,σ) be a metric-like space. A sequence $\{u_n\}$ in Y converges to a point $u \in Y$, with respect to τ_σ, if and only if $\sigma(u,u) = \lim_{n\to\infty} \sigma(u,u_n)$.

Definition 1.5 [4] Let (Y,σ) be a metric-like space.

(a) A sequence $\{u_n\}$ in Y is a Cauchy sequence whenever $\lim_{n,m\to\infty} \sigma(u_n, u_m)$ exists and is finite.

(b) (Y,σ) is complete if every Cauchy sequence $\{u_n\}$ in Y converges with respect to τ_σ to a point $u \in Y$.

That is, $\lim_{n\to\infty} \sigma(u,u_n) = \sigma(u,u) = \lim_{m,n\to\infty} \sigma(u_n, u_m)$.

Definition 1.6 [4] Let S be a self-map defined on metric-like space (Y,σ). Let $\{u_n\}$ be a sequence in Y such that $\sigma(u_n,u) \to \sigma(u,u)$ as $n \to \infty$, we have $\sigma(Su_n,Su) \to \sigma(Su,Su)$ as $n \to \infty$. Then the mapping S is continuous.

Lemma 1.7 [11] Let (Y, σ) be a metric-like space and $\{u_n\}$ be a sequence in Y such that

$u_n \to u$, where $u \in Y$ and $\sigma(u, u) = 0$.

Then,

$\lim_{n\to\infty} \sigma(u_n, v) = \sigma(u, v)$, for all $v \in Y$.

Let Φ be the family of functions $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the conditions:

(i) ϕ is non-decreasing;

(ii) $\sum_{n=1}^{+\infty} \phi^n(t) < \infty$, for all $t > 0$.

Also, if $\phi \in \Phi$, we have $\phi(t) < t$, for all $t > 0$.

The class of $\alpha -$ admissible mappings were introduced by Samet *et al.* [15] in 2012.

Definition 1.8 [15] Let S be a self-map defined on any non-empty set Y and $\alpha : Y \times Y \to \mathbb{R}_0^+$ be another mapping. Then S is called $\alpha -$ admissible if

$$\alpha(u, v) \geq 1 \Longrightarrow \alpha(Su, Sv) \geq 1, \text{for all } u, v \in Y. \quad \textbf{(1)}$$

The concept of $\alpha - \phi$ contractive mapping is also defined in the following way.

Definition 1.9 [15] Let S be a self-map defined on a metric-space (Y, d). We say that S is $\alpha - \phi$ contractive mapping if there exist two functions $\alpha : Y \times Y \to \mathbb{R}_0^+$ and $\phi \in \Psi$ such that

$$\alpha(u, v) d(Su, Sv) \leq \phi\big(d(u, v)\big), \text{for all } u, v \in Y. \quad \textbf{(2)}$$

Some writers have proved fixed point results using the function α for generalized contractions see for example ([3], [6], [7], [10]). Now, we state a generalization of the $\alpha - \phi$ contractive mapping in the following definition in the context of a metric-like space.

2. Main Results

In this section, we establish the existence of fixed point results for generalized $\alpha - \phi$ contraction in metric-like spaces. Here, examples are provided to prove the validity of our results. Also, an application is used to solve boundary value problems for second order differential equation.

Definition 2.1 Let S be a self-map defined on metric-like space (Y, σ). If there exist two functions $\alpha : Y \times Y \to \mathbb{R}_0^+$ and $\phi \in \Phi$ such that

$$\alpha(u, v)\sigma(Su, Sv) \leq \phi\big(M_1(u, v)\big), \text{for all } u, v \in Y, \quad \textbf{(3)}$$

where

$$M_1(u,v) = max\left\{\begin{matrix} \sigma(u,v), \sigma(u,Su), \sigma(v,Sv), \\ \frac{\sigma(v,Su)+\sigma(u,Sv)}{2}, \frac{\sigma(u,Su)+\sigma(v,Sv)}{2} \\ , \frac{[1+\sigma(u,Su)]\sigma(v,Sv)}{\sigma(u,v)+1}, \frac{[1+\sigma(v,Sv)]\sigma(u,Su)}{\sigma(u,v)+1} \end{matrix}\right\}. \quad \textbf{(4)}$$

Then, we say that S is a generalized $\alpha - \phi$ contractive mapping of type A.

Theorem 2.2 Let S be a self-map defined on complete metric-like space (Y, σ) and a generalized $\alpha - \phi$ contractive mapping of type A. Consider that

(i) S is $\alpha -$ admissible;

(ii) There exists $u_0 \in Y$ such that $\alpha(u_0, Su_0) \geq 1$;

(iii) S is continuous.

Then, there exists a point $\xi \in Y$ such that $\sigma(\xi, \xi) = 0$. In addition, suppose that

(HY1) If $\sigma(u,u) = 0$ for some $u \in Y$, then $\alpha(u,u) \geq 1$.

Then such ξ is a fixed point of S.

Proof. By assumption (ii), there exists a point $u_0 \in Y$ such that

$$\alpha(u_0, Su_0) \geq 1.$$

Define a sequence $\{u_n\}$ in Y by $u_{n+1} = Su_n = S^{n+1}u_0$, for all $n \geq 0$.

Assume that $u_{n_0} = u_{n_0+1}$, for some n_0.

Therefore the proof is done, since $\xi = u_{n_0} = u_{n_0+1} = Su_{n_0} = S\xi$.

Also, consider $u_n \neq u_{n+1}$, for all n. **(5)**

Now, $\alpha(u_0, u_1) = \alpha(u_0, Su_0) \geq 1$.

That is, $\alpha(Su_0, Su_1) = \alpha(u_1, u_2) \geq 1$,

Since S is $\alpha -$ admissible.

Repeat the above process, we conclude that

$$\alpha(u_n, u_{n+1}) \geq 1, \text{ for all } n = 0, 1, \ldots \quad \textbf{(6)}$$

We shall prove that $\lim_{n\to\infty} \sigma(u_n, u_{n+1}) = 0$. **(7)**

Combining equations (3) and (6), we get

$\sigma(u_n, u_{n+1}) = \sigma(Su_{n-1}, Su_n)$

$\leq \alpha(u_{n-1}, u_n)\sigma(Su_{n-1}, Su_n)$

$\leq \phi\big(M_1(u_{n-1}, u_n)\big)$, for all $n \geq 1$, **(8)**

where

$$M_1(u_{n-1},u_n) = max\left\{\begin{array}{c}\sigma(u_{n-1},u_n),\sigma(u_{n-1},Su_{n-1}),\sigma(u_n,Su_n)\\ ,\dfrac{\sigma(u_n,Su_{n-1})+\sigma(u_{n-1},Su_n)}{2},\\ \dfrac{\sigma(u_{n-1},Su_{n-1})+\sigma(u_n,Su_n)}{2},\\ \dfrac{[1+\sigma(u_{n-1},Su_{n-1})]\sigma(u_n,Su_n)}{\sigma(u_{n-1},u_n)+1},\\ \dfrac{[1+\sigma(u_n,Su_n)]\sigma(u_{n-1},Su_{n-1})}{\sigma(u_{n-1},u_n)+1}\end{array}\right\}$$

$$= max\left\{\begin{array}{c}\sigma(u_{n-1},u_n),\sigma(u_{n-1},u_n),\sigma(u_n,u_{n+1}),\\ \dfrac{\sigma(u_n,u_n)+\sigma(u_{n-1},u_{n+1})}{2},\\ \dfrac{\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1})}{2},\\ \dfrac{[1+\sigma(u_{n-1},u_n)]\sigma(u_n,u_{n+1})}{\sigma(u_{n-1},u_n)+1},\\ \dfrac{[1+\sigma(u_n,u_{n+1})]\sigma(u_{n-1},u_n)}{\sigma(u_{n-1},u_n)+1}\end{array}\right\}$$

$$\leq max\left\{\begin{array}{c}\sigma(u_{n-1},u_n),\sigma(u_{n-1},u_n),\sigma(u_n,u_{n+1}),\\ \dfrac{\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1})}{2},\\ \dfrac{\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1})}{2},\\ \sigma(u_n,u_{n+1}),\\ \dfrac{[1+\sigma(u_n,u_{n+1})]\sigma(u_{n-1},u_n)}{\sigma(u_{n-1},u_n)+1}\end{array}\right\}$$

$$= max\{\sigma(u_{n-1},u_n),\sigma(u_n,u_{n+1})\}\,. \quad \textbf{(9)}$$

If $max\{\sigma(u_{n-1},u_n),\sigma(u_n,u_{n+1})\} = \sigma(u_n,u_{n+1}) \neq 0$ for some n,

Then from equations (8) and (9), we get

$$\sigma(u_n,u_{n+1}) \leq \phi\big(M_1(u_{n-1},u_n)\big)$$

a contradiction.

Hence,

$$max\{\sigma(u_{n-1},u_n),\sigma(u_n,u_{n+1})\} = \sigma(u_{n-1},u_n),$$

for all $n \in \mathbb{N}$ and equation (8) becomes

$$\sigma(u_n,u_{n+1}) \leq \phi\big(\sigma(u_{n-1},u_n)\big), \quad \textbf{(10)}$$

that is,

$$\begin{aligned}\sigma(u_n,u_{n+1}) &\leq \phi\big(\sigma(u_{n-1},u_n)\big)\\ &< \sigma(u_{n-1},u_n),\end{aligned} \quad \textbf{(11)}$$

for all $n \in \mathbb{N}$.

From equation (10), we get

$$\sigma(u_n, u_{n+1}) \leq \phi^n\big(\sigma(u_0, u_1)\big), \tag{12}$$

for all $n \in \mathbb{N}$.

By properties of ϕ, we have

$$\lim_{n\to\infty} \sigma(u_n, u_{n+1}) = 0.$$

Now, we will show that $\{u_n\}$ is a Cauchy sequence.

Firstly, using (σ_3) and equation (12), we get

$$\sigma(u_n, u_{n+k}) \leq \sigma(u_n, u_{n+1}) + \sigma(u_{n+1}, u_{n+2}) + \cdots + \sigma(u_{n+k-1}, u_{n+k})$$

$$\leq \sum_{p=n}^{n+k-1} \phi^p\left(\sigma(u_0, u_1)\right) \tag{13}$$

$$\leq \sum_{p=n}^{\infty} \phi^p\left(\sigma(u_0, u_1)\right) \to 0 \text{ as } n \to \infty.$$

Thus, by the symmetry of σ, we obtain

$$\lim_{n,m\to\infty} \sigma(u_n, u_m) = 0. \tag{14}$$

Therefore, we conclude that $\{u_n\}$ is a Cauchy sequence in (Y, σ), because (Y, σ) is complete, then there exists a point $\xi \in Y$ such that

$$\lim_{n\to\infty} \sigma(u_n, \xi) = \sigma(\xi, \xi) = \lim_{n,m\to\infty} \sigma(u_n, u_m) = 0. \tag{15}$$

Since S is continuous, from equation (15), we get

$$\lim_{n\to\infty} \sigma(u_{n+1}, S\xi) = \lim_{n\to\infty} \sigma(Su_n, S\xi) = \sigma(S\xi, S\xi). \tag{16}$$

Also, by using equation (15) and Lemma 1.7, we have

$$\lim_{n\to\infty} \sigma(u_{n+1}, S\xi) = \sigma(\xi, S\xi). \tag{17}$$

From equations (16) and (17), we get

$$\sigma(\xi, S\xi) = \sigma(S\xi, S\xi).$$

By equation (3),

$$\alpha(\xi, \xi)\sigma(S\xi, S\xi) \leq \phi\big(M_1(\xi, \xi)\big),$$

where

$$M_1(\xi,\xi) = max\left\{\begin{matrix} \sigma(\xi,\xi), \sigma(\xi,S\xi), \sigma(\xi,S\xi), \\ \dfrac{\sigma(\xi,S\xi)+\sigma(\xi,S\xi)}{2}, \dfrac{\sigma(\xi,S\xi)+\sigma(\xi,S\xi)}{2} \\ , \dfrac{[1+\sigma(\xi,S\xi)]\sigma(\xi,S\xi)}{\sigma(\xi,\xi)+1}, \dfrac{[1+\sigma(\xi,S\xi)]\sigma(\xi,S\xi)}{\sigma(\xi,\xi)+1} \end{matrix}\right\}$$

$$= max\{0, \sigma(\xi, S\xi)\}$$

$= \sigma(\xi, S\xi)$.

Now, from hypothesis (HY1) and fact that $\sigma(\xi, \xi) = 0$, we have

$\alpha(\xi, \xi) \geq 1$.

Hence, from equation (3)

$\sigma(\xi, S\xi) \leq \alpha(\xi, \xi)\sigma(\xi, S\xi)$

$\leq \phi\big(\sigma(\xi, S\xi)\big)$,

which holds unless $\sigma(\xi, S\xi) = 0$.

That is, $\xi = S\xi$.

Therefore, we get ξ is a fixed point of S.

Remark 2.3 If we change the hypothesis of continuity by the following property, then Theorem 2.2 remains true. This implies that there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\alpha(u_{n(k)}, u) \geq 1$ for all k, if $\{u_n\}$ is a sequence in Y such that $\alpha(u_n, u_{n+1}) \geq 1$ for all n and $u_n \to u \in Y$ as $n \to \infty$.

Theorem 2.4 Let S be a self-map defined on complete metric-like space (Y, σ) and a generalized $\alpha - \phi$ contractive mapping of type A. Consider that

(i) S is $\alpha -$ admissible;

(ii) There exists $u_0 \in Y$ such that $\alpha(u_0, Su_0) \geq 1$;

(iii) There exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\alpha\big(u_{n(k)}, u\big) \geq 1$ for all k, if $\{u_n\}$ is a sequence in Y such that $\alpha(u_n, u_{n+1}) \geq 1$ for all n and $u_n \to u \in Y$ as $n \to \infty$.

Then there exists $\xi \in Y$ such that $S\xi = \xi$.

Proof. Following the above proof of Theorem 2.2, we know that the sequence $\{u_n\}$ is defined by $u_{n+1} = Su_n$ for all $n \geq 0$, which is a Cauchy sequence in (Y, σ) and converges to some $\xi \in Y$.

Also, equation (15) holds, so

$$\lim_{k\to\infty} \sigma(u_{n(k)+1}, S\xi) = \sigma(\xi, S\xi). \quad \textbf{(18)}$$

Now, we will prove that $S\xi = \xi$.

Let us consider that $S\xi \neq \xi$ that is, $\sigma(S\xi, \xi) > 0$.

From equation (6) and condition (iii), there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that

$\alpha\big(u_{n(k)}, l\big) \geq 1$, for all k.

By applying equation (3), we get

$\sigma\left(u_{n(k)+1}, S\xi\right) \le \alpha\left(u_{n(k)}, \xi\right)\sigma\left(Su_{n(k)}, S\xi\right)$

$\le \phi\left(M_1\left(u_{n(k)}, \xi\right)\right),$ **(19)**

where

$$M_1\left(u_{n(k)}, \xi\right) = max \begin{Bmatrix} \sigma\left(u_{n(k)}, \xi\right), \sigma\left(u_{n(k)}, Su_{n(k)}\right), \sigma(\xi, S\xi), \\ \frac{\sigma\left(\xi, Su_{n(k)}\right) + \sigma\left(u_{n(k)}, S\xi\right)}{2}, \\ \frac{\sigma\left(u_{n(k)}, Su_{n(k)}\right) + \sigma(\xi, S\xi)}{2}, \\ \frac{\left[1 + \sigma\left(u_{n(k)}, Su_{n(k)}\right)\right]\sigma(\xi, S\xi)}{\sigma\left(u_{n(k)}, \xi\right) + 1}, \\ \frac{[1 + \sigma(\xi, S\xi)]\sigma(u_{n(k)}, Su_{n(k)})}{\sigma\left(u_{n(k)}, \xi\right) + 1} \end{Bmatrix}$$

$$= max \begin{Bmatrix} \sigma\left(u_{n(k)}, \xi\right), \sigma\left(u_{n(k)}, u_{n(k)+1}\right), \sigma(\xi, S\xi), \\ \frac{\sigma\left(\xi, u_{n(k)+1}\right) + \sigma\left(u_{n(k)}, S\xi\right)}{2}, \\ \frac{\sigma\left(u_{n(k)}, u_{n(k)+1}\right) + \sigma(\xi, S\xi)}{2}, \\ \frac{\left[1 + \sigma\left(u_{n(k)}, u_{n(k)+1}\right)\right]\sigma(\xi, S\xi)}{\sigma\left(u_{n(k)}, \xi\right) + 1}, \\ \frac{[1 + \sigma(\xi, S\xi)]\sigma\left(u_{n(k)}, u_{n(k)+1}\right)}{\sigma\left(u_{n(k)}, \xi\right) + 1} \end{Bmatrix}. \quad \textbf{(20)}$$

By equations (7) and (18), we obtained

$\lim_{k\to\infty} M_1(u_{n(k)}, \xi) = \sigma(\xi, S\xi).$ **(21)**

Letting $k \to \infty$ in equation (19), we have

$\sigma(\xi, S\xi) \le \phi\left(\sigma(\xi, S\xi)\right) < \sigma(\xi, S\xi),$ **(22)**

a contradiction.

Hence, ξ is a fixed point of S, that is, $S\xi = \xi$.

Definition 2.5 Let S be a self-map defined on metric-like space (Y, σ). If there exist two functions $\alpha : Y \times Y \to \mathbb{R}_0^+$ and $\phi \in \Phi$ such that

$$\alpha(u, v)\sigma(Su, Sv) \le \phi\left(M_2(u, v)\right), \text{for all } u, v \in Y, \quad \textbf{(23)}$$

where

$$M_2(u, v) = max \begin{Bmatrix} \sigma(u, v), \sigma(u, Su), \sigma(v, Sv), \\ \frac{[1+\sigma(u,Su)]\sigma(v,Sv)}{\sigma(u,v)+1} \end{Bmatrix}. \quad \textbf{(24)}$$

Then, we say that S is a generalized $\alpha-\phi$ contractive mapping of type B.

Theorem 2.6 Let S be a self-map defined on complete metric-like space (Y,σ) and a generalized $\alpha-\phi$ contractive mapping of type B. Consider that

(i) S is $\alpha-$ admissible;

(ii) There exists $u_0 \in Y$ such that $\alpha(u_0, Su_0) \geq 1$;

(iii) S is continuous.

Then, there exists $\xi \in Y$ such that $\sigma(\xi,\xi)=0$. If in addition (HY1) holds, then such ξ is a fixed point of S.

Proof. Along the lines of the proof of Theorem 2.2, we obtain the required result. as of the analogy, we skip the explanation of the proof.

Theorem 2.7 Let S be a self-map defined on complete metric-like space (Y,σ) and a generalized $\alpha-\phi$ contractive mapping of type B. Suppose that

(i) S is $\alpha-$ admissible;

(ii) There exists $u_0 \in Y$ such that $\alpha(u_0, Su_0) \geq 1$;

(iii) There exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\alpha(u_{n(k)}, u) \geq 1$ for all k, if $\{u_n\}$ is a sequence in Y such that $\alpha(u_n, u_{n+1}) \geq 1$, for all n and $u_n \to u \in Y$ as $n \to \infty$.

Then, there exists $\xi \in Y$ such that $S\xi = \xi$.

Due to the similarity of Theorem 2.4, we omit the proof.

Corollary 2.8 Let S be a self-map defined on complete metric-like space (Y,σ) such that

$\sigma(Su,Sv) \leq \phi(M_1(u,v))$, for all $u,v \in Y$,

where

$$M_1(u,v) = max\left\{\begin{matrix} \sigma(u,v), \sigma(u,Su), \sigma(v,Sv), \\ \dfrac{\sigma(v,Su)+\sigma(u,Sv)}{2}, \dfrac{\sigma(u,Su)+\sigma(v,Sv)}{2} \\ , \dfrac{[1+\sigma(u,Su)]\sigma(v,Sv)}{\sigma(u,v)+1}, \dfrac{[1+\sigma(v,Sv)]\sigma(u,Su)}{\sigma(u,v)+1} \end{matrix}\right\}.$$

Then, S has a fixed point.

Proof. If $\alpha(u,v)=1$ in Theorem 2.4, then the result is proved.

Corollary 2.9 Let S be a self-map defined on complete metric-like space (Y,σ) such that

$\sigma(Su,Sv) \leq \lambda\, M_1(u,v)$, for all $u,v \in Y$,

where $\lambda \in [0,1)$.

Then, S has a fixed point.

Proof. If $\phi(t) = \lambda t$ in Corollary 2.8, then the result is proved.

Corollary 2.10 Let S be a self-map defined on complete metric-like space (Y,σ) such that

$\sigma(Su,Sv) \leq \phi\big(M_2(u,v)\big)$, for all $u,v \in Y$,

where

$$M_2(u,v) = max\left\{\begin{matrix}\sigma(u,v),\sigma(u,Su),\sigma(v,Sv),\\ \dfrac{[1+\sigma(u,Su)]\sigma(v,Sv)}{\sigma(u,v)+1}\end{matrix}\right\}.$$

Then, S has a fixed point.

Proof. If $\alpha(u,v) = 1$ in Theorem 2.6, then the result is proved.

Corollary 2.11 Let S be a self-map defined on complete metric-like space (Y,σ) such that

$\sigma(Su,Sv) \leq \lambda\, M_2(u,v)$, for all $u,v \in Y$,

where $\lambda \in [0,1)$.

Then, S has a fixed point.

Proof. If $\phi(t) = \lambda t$ in Corollary 2.10, then the result is proved.

3. Examples

Here, we give two solid examples to support our results.

Example 3.1 Consider $Y = \{0,2,3\}$. Take metric-like (dislocated) $\sigma : Y \times Y \to \mathbb{R}_0^+$ defined by

$\sigma(0,0) = \sigma(2,2) = 0, \sigma(3,3) = \frac{8}{20},$

$\sigma(0,3) = \sigma(3,0) = \frac{1}{3}, \sigma(2,3) = \sigma(3,2) = \frac{1}{2},$

$\sigma(0,2) = \sigma(2,0) = \frac{2}{5}.$

Here, $\sigma(3,3) \neq 0$, so σ is not a metric and $\sigma(3,3) > \sigma(0,3)$, so σ is not a partial metric.

Clearly, (Y, σ) is a complete metric-like space.

Given $S : Y \to Y$ as $S0 = S2 = 0$ and $S3 = 2$.

Take $\phi(t) = \frac{4}{5}t$ for each $t \geq 0$.x

Define the mapping $\alpha : Y \times Y \to [0, \infty)$ by

$$\alpha(u, v) = \begin{cases} 1 & \text{if } u = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Firstly, suppose that $u, v \in Y$ such that $\alpha(u, v) \geq 1$.

Now, $u = 0$ by the definition of α and since $S0 = 0$, so $\alpha(Su, Sv) = 1$ for each $v \in Y$, this implies that S is $\alpha -$ admissible.

Now, we will consider the following cases:

Case 1: If $(u = 0$ and $v = 0)$ or $(u = 0$ and $v = 2)$ or $(u = 2$ and $v = 2)$, we have

$\alpha(Su, Sv)\sigma(Su, Sv) = \sigma(Su, Sv) = 0.$

Case 2: If $(u = 3$ and $v = 3)$, we have

$\alpha(Su, Sv)\sigma(Su, Sv) = 0.$

Case 3: If $(u = 0$ and $v = 3)$ or $(u = 2$ or $v = 3)$, we have

$$\alpha(Su, Sv)\sigma(Su, Sv) = \sigma(Su, Sv) = \sigma(0,2) = \frac{2}{5} = \frac{4}{5}\sigma(3,2) = \phi\big(\sigma(v, Sv)\big) \leq \phi\big(M_1(u, v)\big),$$

where $M_1(u, v)$ is defined by equation (4),

Hypothesis (iii) of Theorem 2.4 is satisfied. Thus, we may apply Theorem 2.4 and so S has a fixed point, that is, $\xi = 0$.

Example 3.2 Let $Y = \mathbb{R}_0^+$ be with metric-like σ defined as $\sigma(u, v) = \max\{u, v\}$.

Define the mapping $S : Y \to Y$ by

$$Su = \begin{cases} \frac{1}{3}u^3 \text{ if } u \in \left[0, \frac{1}{2}\right], \\ 2u - 1 \text{ otherwise.} \end{cases}$$

Also, consider $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ given as

$$\phi(t) = \begin{cases} \frac{1}{3}t^3 & \text{if } 0 \leq t < \frac{1}{2} \\ \frac{1}{3} & \text{otherwise.} \end{cases}$$

Clearly $\phi \in \Phi$. Now, assume that $\alpha : Y \times Y \to \mathbb{R}_0^+$ given as

$$\alpha(u,v) = \begin{cases} 1 \text{ if } u,v \in \left[0,\frac{1}{2}\right] \\ 0 \text{ otherwise.} \end{cases}$$

and,

$$M_1(u,v) = max \begin{Bmatrix} \sigma(u,v), \sigma(u,Su), \sigma(v,Sv), \\ \frac{\sigma(v,Su)+\sigma(u,Sv)}{2}, \frac{\sigma(u,Su)+\sigma(v,Sv)}{2}, \\ \frac{[1+\sigma(u,Su)]\sigma(v,Sv)}{\sigma(u,v)+1}, \frac{[1+\sigma(v,Sv)]\sigma(u,Su)}{\sigma(u,v)+1} \end{Bmatrix}.$$

Firstly, let $u, v \in Y$ such that $\alpha(u,v) \geq 1$, so $u, v \in \left[0,\frac{1}{2}\right]$.

In this case, $\alpha(Su,Sv) = \alpha\left(\frac{1}{3}u^3, \frac{1}{3}v^3\right) = 1$;

That is, S is $\alpha -$ admissible.

Also,

$\alpha(Su,Sv)\sigma(Su,Sv) = \sigma(Su,Sv)$

$= \sigma\left(\frac{1}{3}u^3, \frac{1}{3}v^3\right)$

$= \sigma\big(\phi(u), \phi(v)\big)$

$= \max\{\phi(u), \phi(v)\}$

$= \phi(\max\{u,v\})$

$= \phi\big(\sigma(u,v)\big)$

$\leq \phi\big(M_1(u,v)\big).$

Now, hypothesis (iii) of Theorem 2.4 is also satisfied. By applying Theorem 2.4, S have two fixed points in Y, which are $\xi = 0$ and $\xi = 1$.

4. Applications

We assume two-point boundary-value problem for the second order differential equation

$-\frac{d^2u}{dt^2} = f\big(t, u(t)\big), t \in [0,1]$

$u(0) = u(1) = 0,$ **(25)**

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Also, Green's function associated to equation (25) is

$$G(t,r) = \begin{cases} t(1-r) \; if \; 0 \leq t \leq r \leq 1 \\ r(1-t) \; if \; 0 \leq r \leq t \leq 1. \end{cases} \quad \textbf{(26)}$$

Let $Y = C(I), (I = [0,1])$ be the set of real continuous functions defined on I and $\sigma(f,g) = sup_{t\in[0,1]}(|f(t)| + |g(t)|)$, for all $f, g \in C(I,\mathbb{R})$.

It is well known that $u \in C^2(I)$ is a solution of equation (25) is equivalent to that $u \in Y = C(I,\mathbb{R})$ is a solution of the integral equation

$$u(t) = \int_0^1 G(t,r) f\big(r, u(r)\big) dr \text{ , for all } t \in I. \quad \textbf{(27)}$$

Theorem 4.1 Suppose that these conditions hold:

(i) There exists a continuous function $p : I \to \mathbb{R}_0^+$ such that

$|f(r,c)| \leq 8p(r)|c|,$

for each $r \in I$ and $c \in \mathbb{R}$;

(ii) There exists a continuous function $q : I \to \mathbb{R}_0^+$ such that

$|f(r,d)| \leq 8q(r)|d|,$

for each $r \in I$ and $d \in \mathbb{R}$;

(iii) $sup_{r\in I} p(r) = \lambda_1 < \frac{1}{2}$;

(iv) $sup_{r\in I} q(r) = \lambda_2 < \frac{1}{2}$.

Then, problem (25) has a solution $\xi \in Y = C(I,\mathbb{R})$.

Proof. Consider the mapping $S : Y \to Y$ defined by

$Su(t) = \int_0^1 G(t,r) f\big(r, u(r)\big) dr,$

for all $u \in Y$ and $t \in I$.

Here, problem (25) is equivalent to find $\xi \in Y$ that is a fixed point of S.

Let $u, v \in Y$, we get

$$|Su(t)| = \left| \int_0^1 G(t,r) f\big(r,u(r)\big) dr \right|$$

$$\leq \int_0^1 G(t,r) \big| f\big(r,u(r)\big) \big| dr$$

$$\leq 8 \int_0^1 G(t,r) p(r) |u(r)| dr.$$

Taking sup on both sides, we have

$sup_{t\in[0,1]}|Su(t)| \leq 8\lambda_1 sup_{t\in[0,1]}|u(t)|sup_{t\in[0,1]} \int_0^1 G(t,r)dr$

$= \lambda_1 sup_{t\in[0,1]}|u(t)|$.

As $\int_0^1 G(t,r)dr = \frac{-t^2}{2} + \frac{t}{2}$,

And so $sup_{t\in[0,1]} \int_0^1 G(t,r)dr = \frac{1}{8}$.

Similarly,

$|Sv(t)| = \left|\int_0^1 G(t,r)f(r,v(r))dr\right|$

$\leq \int_0^1 G(t,r)\left|f(r,v(r))\right|dr$

$\leq 8\int_0^1 G(t,r)q(r)|v(r)|dr.$

Taking sup on both sides, we have

$sup_{t\in[0,1]}|Sv(t)| \leq 8\lambda_2 sup_{t\in[0,1]}|v(t)|sup_{t\in[0,1]} \int_0^1 G(t,r)dr$

$= \lambda_2 sup_{t\in[0,1]}|v(t)|$.

As $\int_0^1 G(t,r)dr = \frac{-t^2}{2} + \frac{t}{2}$,

And, so $sup_{t\in[0,1]} \int_0^1 G(t,r)dr = \frac{1}{8}$.

Take $\lambda = \lambda_1 + \lambda_2$. Using assumptions of Theorem 4.1, we get $\lambda < 1$.

Also, $\sigma(Su,Sv) = sup_{t\in[0,1]}(|Su(t)| + |Sv(t)|)$

$\leq \lambda_1 sup_{t\in[0,1]}|u(t)| + \lambda_2 sup_{t\in[0,1]}|v(t)|$

$\leq (\lambda_1 + \lambda_2) sup_{t\in[0,1]}(|u(t)| + |v(t)|)$

$= \lambda sup_{t\in[0,1]}(|u(t)| + |v(t)|)$

$= \lambda\sigma(u,v)$.

Therefore, $\sigma(Su,Sv) \leq \lambda\sigma(u,v)$.

Hence, all assumptions of Corollary 2.9 are satisfied, so we can say that S has a fixed point $\xi \in Y$, That is, the problem (25) has a solution $\xi \in Y$.

References

[1] Aage C. T., Salunke J. N., *Some results of fixed point theorem in dislocated quasi metric space*, Bull. Marathadawa Math. Soc., **9**(2008), 1-5.

[2] Aage C. T., Salunke J. N., *The results of fixed points in dislocated and dislocated

quasi metric space, Appl. Math. Sci., **2**(2008), 2941-2948.

[3] Ali M. U., Kamran T., *On* $(\alpha^*, \psi) -$ *contractive multi-valued mappings*, Fixed Point Theory Appl., **2013**(2013).

[4] Harandi A., *Metric-like spaces, partial metric spaces and fixed points*, Fixed Point Theory Appl., **2012**(2012), 204.

[5] Banach S., *Sur les operations dans les ensembles abstraits et leur application aux equations integrals*, Fund. Math., **3**(1922), 133-181.

[6] Aydi H., Jellali M., Karapinar E., *Common fixed points for generalized* α*-implicit contractions in partial metric spaces: Consequences and application*, RACSAM, (2014), Doi: 10.1007/s13398-014-0187-1.

[7] Aydi H., Jellali M., Karapinar E., *On fixed point results for* α*-implicit contractions in quasi metric spaces and consequences*, Accepted in Nonlinear Analysis: Modelling and Control, (2014).

[8] Isufati A., *Fixed point theorem in dislocated quasi metric spaces*, Appl. Math. Sci., **4**(2010), 217-223.

[9] Hitzler P., Seda A. K., *Dislocated topologies*, J. Electr. Engin., **51**(2000), 3-7.

[10] Karapinar E., Samet B., *Generalized* $(\alpha - \psi)$ *contractive type mappings and related fixed point theorems with applications*, Abstr. Appl. Anal., **2012**(2012), Article id:793486.

[11] Karapinar E., Salimi P., *Dislocated metric space to metric spaces with some fixed point theorems*, Fixed Point Theory Appl., **2013**(2013), 222.

[12] Kohli M., Shrivastava R., Sharma M., *Some results on fixed point theorems in dislocated quasi metric space*, Int. J. Theoret. Appl. Sci., **2**(2010), 27-28.

[13] Matthews S. G., *Partial metric topology*, *In Proceedings of the 8th Summer Conference on General Topology and Applications*, Ann. New York Acad. Sci., **728**(1994), 183-197.

[14] Ren Y., Li J., Yu Y., *Common fixed point theorems for non-linear contractive mappings in dislocated metric spaces*, Abstract and Applied Analysis, Volume 2013, Article ID 483059, 5 pages.

[15] Samet B., Vetro C., Vetro P., *Fixed point theorems for* $(\alpha - \psi)$ *contractive type mappings*, Non-linear Anal., **75**(2012), 2154-2165.

[16] Zeyada Z. M., Hassan G. H., Ahmad M. A., *A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi metric space*, Arab. J. Sci. Eng., **31**(2005), 111-114.

[17] Zoto K., Housxha E., Isufati A., *Some new results in dislocated and dislocated quasi metric space*, Appl. Math. Sci., **6**(2012), 3519-3526.