Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 20, Number 2 (2024), pp. 331-336

http://www.ripublication.com

Construction of New Extra Loops from Old One

[1]David Joel Mengue Mengue* and [2]Alexandre Fotue Tabue

[1]University of Ebolowa, Cameroon

[2]University of Bertoua, Cameroon

Abstract

Extra loops are varieties of loops of Bol-Moufang type. [9] show that there are exactly 14 such varieties. Their identities is defined such that if two of its three variables occurs once on each side the third variable occurs twice on each side, and the order in which the variable appear on both sides is the same. In this paper we construct new extra loops from old exra loops.

INTRODUCTION

Extra loops are loops defined with three different equations .For each of them the identity $\alpha = \beta$ is said to be of Bol-Moufang type if a-) the only operation in the $\alpha = \beta$ is a binary operation. b-) the number of distinct variables appearing in α (and thus inβ) is 3, c-) the number of variables appearing in α (and thus in β) is 4, d-) the order in which the variables appear in α coincides with the order in which they appear in β. Let remind the reader that: letr $(G,\bullet)$ be a groupoid. To every element $a \in G$ is associated with bijective mappings $L(a)$ and $R(a)$ of G onto itself. These mappings are called left and right translations. $(G, \bullet)$ with these translations is called quasigroup.

Definition 1.1 Let $(G,\bullet)$ be a quasigroup, if there exist an element e in G such that for all $x \in G \;\; x \bullet e = e \bullet x = x$ then e is said to be the identity element of $(G,\bullet)$. A quasigroup with an identity element is called a loop.

In [7] theorem I.1.7 Hala Pflugfelder showed that if $(G,\bullet)$ is an association quasigroup then it has an identity element and hence it becomes a group. In [7] hala Pflufelder gave an example (I.1.8) which is a quasigroup with identity but not associative. This shows that quasigroup with identity element need not be associative. Thus loop do exist.

Definition 1.2 let $(G,\bullet)$ be a loop and satisfies any of the following identities:

i) $((a \bullet b) \bullet c) \bullet a = a \bullet (b \bullet (c \bullet a))$

ii) $(a \bullet b) \bullet (a \bullet c) = a \bullet ((b \bullet a) \bullet c)$

iii) $(b \bullet a) \bullet (c \bullet a) = (b \bullet (a \bullet c)) \bullet a$

For all $a, b, c \in G$ is called an extra loop.

In [2], Fenyves proved that the identities i), ii) and iii) are equivalent. It then follows that to show that a given loop is an extra loop, we only need to show that it satisfies any of the three identities stated above.

Any of the above identity confirm that an extra loops is a variety of loops of Bol-Moufang type

Definition 1.3 Let $(G,\bullet)$ be a quasigroup, from this quasigroup, we define five other quasigroups $(G,\circ)$, $(G,*)$, $(G,\odot)$, $(G,\backslash)$ and $(G,/)$ which are called parastrophs or conjugates of $(G,\bullet)$. Each of the operation on G is related to the operation • on G as follows:

i) $a \circ b = c$ means that $b \bullet a = c$

ii) $a * b = c$ means that $b \bullet c = a$

iii) $a \odot b = c$ means that $b = c \bullet a$

iv) $a \backslash b = c$ means that $b = a \bullet c$

v) $a/b = c$ means that $a = c \bullet b$

The most recent study of parastrophs of a quasigroup are by Sokhatski F.N in [8], by Duplak J in [1], Gushan V.V and Sokhatski in [4] and Jaiyeola Temitope in [5].

2) In this section we assume that a quasigroup $(G,\bullet)$ is an extra loop and we are going to show that each of its parastrophs is also an extra loop.

Lemma 2.1 Let $(G,\bullet)$ be an extra loop and let $(G,\circ)$ be a parastroph of $(G,\bullet)$ then $(G,\circ)$ is also an extra loop where $a \circ b = c$ means that $b \bullet a = c$

Proof: Recall that $(G,\circ)$ is a quasigroup. In [6] Kunen Kenneth showed that $(G,\circ)$ has an identity element hence it is a loop. To show that $(G,\circ)$ is an extra loop, we need to show that it satisfies one of the identities of Definition 1.2. We will show that it satisfies 1.2(i). Recall that $a \circ b = c$ means $b \bullet a = c$ in other words $a \circ b = b \bullet a$. Hence we have for all $a, b, c\ in\ G$;

$$\big((a \circ b) \circ c\big) \circ a = \big((b \bullet a) \circ c\big) \circ a$$

$$= \big(c \bullet (b \bullet a)\big) \circ a$$

$$= a \bullet (c \bullet (b \bullet a))$$

And similarly

$$a \circ \big(b \circ (c \circ a)\big) = a \circ \big(b \circ (a \bullet c)\big)$$
$$= a \circ \big((a \bullet c) \bullet b\big)$$
$$= ((a \bullet c) \bullet b) \bullet a$$

By (i)

$$\big((a \bullet b) \bullet c\big) \bullet a = a \bullet (b \bullet (c \bullet a))$$

It then follows that

$$\big((a \circ b) \circ c\big) \circ a = a \circ (b \circ (c \circ a))$$

Hence $(G,\circ)$ is an extra loop.

Lemma 2.2 Let $(G,\bullet)$ be a loop with identity element e and $\forall a \in G \quad a^2 = e$ then $(G,\bullet)$ has the automorphic inverse property.

Proof Since for all $a\ in\ G\ \ a^2 = e$ it follows that $a^{-1} = a$, now for all $a, b\ in\ G$ we have $(a \bullet b)^2 = e$ this implies $(a \bullet b) \bullet (a \bullet b) = e$ and so $a \bullet b = (a \bullet b)^{-1}$

$$\text{Thus } (a \bullet b)^{-1} = a \bullet b = a^{-1} \bullet b^{-1}$$

This result of lemma 2.2 will be applied in the rest of this paper.

Lemma 2.3 Let $(G,\bullet)$ be an extra loop with identity element e and $(G,\backslash)$ a parastroph of $(G,\bullet)$ where $a \setminus b = c$ means that $a^{-1} \bullet b = c$. If $a^2 = e\ for\ all\ a \in G$ then $(G,\backslash)$ is also an extra loop.

Proof: Since $a^2 = e$ for all $a \in G$ then $a = a^{-1}$ To show that $(G,\backslash)$ is an extra loop, we want to show that $(b \setminus a) \setminus (c \setminus a) = (b \setminus (a \setminus c)) \setminus a$ We have

$$(b \setminus a) \setminus (c \setminus a) = (b^{-1} \bullet a) \setminus (c^{-1} \bullet a)$$
$$= (b^{-1} \bullet a)^{-1} \bullet (c^{-1} \bullet a)$$
$$= (b \bullet a^{-1}) \bullet (c^{-1} \bullet a)$$

Similarly

$$\big(b \setminus (a \setminus c)\big) \setminus a \quad = \big(b \setminus (a^{-1} \bullet c)\big) \setminus a$$
$$= (b^{-1} \bullet (a^{-1} \bullet c)) \setminus a$$
$$= (b^{-1} \bullet (a^{-1} \bullet c))^{-1} \bullet a$$
$$= (b \bullet (a^{-1} \bullet c))^{-1} \bullet a$$
$$= (b \bullet (a \bullet c^{-1})) \bullet a$$

Since $a^{-1} = a$ and $c = c^{-1}$ then it follows that $(b \bullet a^{-1}) \bullet (c^{-1} \bullet a) = (b \bullet a) \bullet (c \bullet a)$ and $\big(b \bullet (a \bullet c^{-1})\big) \bullet a = (b \bullet (a \bullet c)) \bullet a$

Hence $(b \setminus a) \setminus (c \setminus a) = (b \bullet a) \bullet (c \bullet a)$ and $\big(b \setminus (a \setminus c)\big) \setminus a = (b \bullet (a \bullet c)) \bullet a$ by 1.2(iii) $(b \bullet a) \bullet (c \bullet a) = (b \bullet (a \bullet c)) \bullet a$ then $(b \setminus a) \setminus (c \setminus a) = (b \setminus (a \setminus c)) \setminus a$ hence $(G,\setminus)$ is an extra loop.

Lemma 2.4 Let $(G,\bullet)$ be an extra loop with identity element e and $(G,*)$ one of it parastroph define by $For\ all\ a, b, c\ in\ G,\ a * b = c\ means\ b^{-1} \bullet a = c$. If $a^2 = e\ for\ all\ a\ in\ G$ then $(G,*)$ is also an extra loop.

Proof: Since $a^2 = e$, then $a^{-1} = a$ For all $a\ in\ G$.

$$\big((a * b) * c\big) * a = \big((b^{-1} \bullet a) * c\big) * a = \big(c^{-1} \bullet (b^{-1} \bullet a)\big) * a$$
$$= a^{-1} \bullet \big(c^{-1} \bullet (b^{-1} \bullet a)\big) = a \bullet (c \bullet (b \bullet a))$$

Similarly

$$a * \big(b * (c * a)\big) = a * \big(b * (a^{-1} \bullet c)\big) = a * ((a^{-1} \bullet c)^{-1} \bullet b)$$
$$= ((a^{-1} \bullet c)^{-1} \bullet b)^{-1} \bullet a = ((a^{-1} \bullet c) \bullet b^{-1}) \bullet a = ((a \bullet c) \bullet b) \bullet a$$

By 1.2(i) $\big((a \bullet c) \bullet b\big) \bullet a = a \bullet (c \bullet (b \bullet a))$

It follows that

$$\big((a * b) * c\big) * a = a * \big(b * (c * a)\big)$$

Hence $(G,*)$ is an extra loop.

Lemma 2.5: Let $(G,\bullet)$ be an extra loop with identity e and $(G,\odot)$ one of it parastroph define by for all $a, b, c\ in\ G$ $a \odot b = c$ means $b \bullet a^{-1} = c$. If for all $x\ in\ G, x^2 = e$ then $(G,\odot)$ is also an extra loop.

Proof: To show that $(G,\odot)$ is an extra loop, we have to show that

$$\big((a \odot b) \odot c\big) \odot a = a \odot (b \odot (c \odot a))$$

Which satisfies 1.2(i)

$$\big((a \odot b) \odot c\big) \odot a = \big((b \bullet a^{-1}) \odot c\big) \odot a$$
$$= (c \bullet (b \bullet a^{-1})^{-1}) \odot a$$
$$= a \bullet (c \bullet (b \bullet a^{-1}))^{-1}$$
$$= a \bullet \big(c^{-1} \bullet (b^{-1} \bullet a)\big)$$
$$= a \bullet (c \bullet (b \bullet a))$$

And similarly

$$a \odot \big(b \odot (c \odot a)\big) = a \odot \big(b \odot (a \bullet c^{-1})\big)$$
$$= a \odot \big((a \bullet c^{-1}) \bullet b^{-1}\big)$$
$$= \big((a \bullet c^{-1}) \bullet b^{-1}\big) \bullet a^{-1}$$

$$= ((a \bullet c) \bullet b) \bullet a$$

By 1.2(i) and $(G,\bullet)$ being an extra loop we have $\big((a \bullet b) \bullet c\big) \bullet a = a \bullet (b \bullet (c \bullet a))$ it then follows that $\big((a \odot b) \odot c\big) \odot a = a \odot (b \odot (c \odot a))$ Hence $(G,\odot)$ is an extra loop.

Lemma 2.6: Let $(G,\bullet)$ be an extra loop which also has an automorphic inverse property and is of exponent two. If $(G,/)$ is one of it parastroph define by: for all $a, b, c \ in\ G \quad a/b = c$ means $a \bullet b^{-1} = c$ then $(G,/)$ is also an extra loop

Proof: To show that $(G,/)$ is an extra loop it will be enough to show that

$$((a/b)/c)/a = a/(b/(c/a))$$

We have

$$((a/b)/c)/a = ((a \bullet b^{-1})/c)/a = ((a \bullet b^{-1}) \bullet c^{-1})/a = ((a \bullet b^{-1}) \bullet c^{-1}) \bullet a^{-1}$$
$$= ((a \bullet b) \bullet c) \bullet a$$

Similarly

$$a/(b\ /\ (c/a)) = a/(b/(c \bullet a^{-1})) = a/(b \bullet (c \bullet a^{-1})^{-1}) = a/(b \bullet (c^{-1} \bullet a))$$
$$= a \bullet (b \bullet (c^{-1} \bullet a))^{-1} = a \bullet (b^{-1} \bullet (c^{-1} \bullet a)^{-1})$$
$$= a \bullet \big(b^{-1} \bullet (c \bullet a^{-1})\big) = a \bullet (b \bullet (c \bullet a))$$

By 1.2(i) we have

$$\big((a \bullet b) \bullet c\big) \bullet a = a \bullet (b \bullet (c \bullet a))$$

It then follows that

$$((a/b)/c)/a = a/(b/(c/a))$$

Hence $(G,/)$ is an extra loop.

Theorem: Let $(G,\bullet)$ be an extra loop which also has automorphic inverse property, let $(G,\circ); (G,*); (G,\odot);\ (G,\backslash) and\ (G,/)$ be its parastrophs as defined in definition 1.3. If $(G,\bullet)$ is of exponent two then each of it parastroph is also an extra loop.

Proof: Combine the proofs of lemma 2.1; 2.2; 2.3; 2. 4; 2.5 et 2.6 all the parastrophs are also extra loops

Another new construction:

For a group G, let $M(G,2) = G \times \{0,1\}$ where $(g,0)(h,0) = (gh,0)$; $(g,0)(h,1) = (hg,1)$; $(g,1)(h,0) = (gh^{-1},1)$ and $(g,1)(h,1) = (h^{-1}g,0)$.

$M(G,2)$ is an extra loop if and only it G is nonabelian

REFERENCES

[1] Duplak J. A parastrophic equivalence in quasigroups and related system (2000) 7-14
[2] Fenyves F. Extra loop I Publ. math Debrecen (1968) 235-238
[3] Fenyves F. Extra loop II Publ. math Debrecen (1969) 187-192
[4] Gushan V.V and Sokhatski K.K. A representation of parastrophs of loops and quasigroups Journal of Discrete math and application (2000) 535-542
[5] Jaiyeola .T.G some necessary and sufficient conditions for Parastrophic equivalence of the associative law in quasigroups. FASCICULI MATHEMATTICI (2008) 25-35
[6] Kunen Kenneth Quasigroups, loops and associative laws Journal of Algebra (1990) 194-204
[7] Mengue Mengue D.J and Ajala S.O. On the concept of derivative of a function in quasigroups and loops theory. Global journal of Mathematics and statistics (2010) 41-47
[8] Pflugfelder H Quasigroups and loops Introduction (1990) Berlin Sigma series in pure Mathematics
[9] Phillips J D and Peer Vojtechoysi. The varieties of loops of Bol-Moufang Type. Algebra Universalis (2005) Vol.54 No.3, pp259-383
[10] Sokhatski F.N On isotopes of a group. Journal Ukrainian math journal (1995) 1585-1598