Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 20, Number 2 (2024), pp. 305-311

http://www.ripublication.com

The entropic quasigroup and its parastrophs

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Abstract

The work we did in this paper is twofold, we first of all came out with some method to construct new entropic quasigroup from old one. Secondly we were able to prove that the parastroph of any entropic quasigroup are also entropic.

INTRODUCTION

Quasigroups and loops theory is a fairly young subject, it is taking its roots from algebra, geometry and combinatory. In literature it is well known that the parastrophs of any quasigroup are all quasigoups but the very same parastroph of a loop are not loops. So what will happen to the parastroph of an entropic quasigroup? Entropic quasigroups are known in mathematical literature under several dfferent names: D.C Murdoch in [Murdoch,5] and R.K Bruck in [Bruck,2([a],[b])] called the as abelian quasigroups, S.K.Stein in [Stein,6] and many other refer to them as medial quasigroup. J.Aczel in [Aczel,1] called this class of quasigroup as “bi-symmetric” and O.Frink in [Frink,4] do call them “symmetric” in this work, we use the name “entropic” which was introduced by I.M.H Etherington in [Etherington,3([a],[b])] The class of entropic quaigroups by no means coincides with the class of distributive quasigroups. But a distributive quasigroup which is entropic is necessarily isotopic to a group. Since there exist distributive quasigroups isotopic to commutative Moufang loops which are not groups, they are distributive quasigroup which are not entropic.

Definition:1 A quasigroup $(Q,\cdot)$ which satisfies the following two identity:

$$x\cdot(y\cdot z)=(x\cdot y)\cdot(x\cdot z)$$

and

$$(x\cdot y)\cdot z=(x\cdot z)\cdot(y\cdot z)$$

is said to be distributive, for all $x,y,z\in Q$

A quasigroup $(Q,\cdot)$ which satisfies the identical relation

$$(x \cdot u) \cdot (v \cdot y) = (x \cdot v) \cdot (u \cdot y) \quad \textbf{(3)}$$

for all $x, u, v, y \in Q$ is called entropic.

But not all entropic quasigroups are idempotent, therefore not all entropic quasigroups are distributive. To support this statement, we give an example of an entropic quasigroup which is not idempotent.

Example:1 Let $Q = \{0,1,2,3,4\}$ and let $(\circ)$ be the operation of substraction modulo 5. Then $(Q,\circ)$ has the following latin square

$\circ$	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

$(Q,\circ)$ is entropic but not idempotent and therefore not distributive.

As contrasted with a distributive quasigroup, an entropic quasigroup which is not distributive may possess a one sided identity element e_ρ or $e_\lambda$or may even be a loop.

If $(Q,\cdot)$ is an entropic loop then $(Q,\cdot)$ is an abelian group.

For the proof, one only need to show associaticity and commutativity laws: for all $x, y, z \in Q$

we have: $x \cdot (y \cdot z) = (x \cdot e) \cdot (y \cdot z) = (x \cdot y) \cdot (e \cdot z) = (x \cdot y) \cdot z$

and

$$x \cdot y = (e \cdot x) \cdot (y \cdot e) = (e \cdot y) \cdot (x \cdot e) = y \cdot x$$

Definition:2 A triple (α, β, γ) of bijection from set G onto a set H is called an isotopism of a groupoid $(G,\cdot)$ onto a groupoid $\{H,\circ)$ provided $(x)\alpha \circ (y)\beta = (x \cdot y)\gamma$ for all $x, y \in G$. $(H,\circ)$ is then called an isotope of $(G,\cdot)$, and groupoids $(G,\cdot)$ and $(H,\circ)$ are called isotope to each other. The concept of isotopy is a generalization of that of isomorphy.

Definition:3 Let α and β be permutations on G and let i denote the identity map on G. Then (α, β, i) is the principal isotopism of a groupoid $(G,\cdot)$ onto a groupoid $(G,\circ)$ means that (α, β, i) is an isotopism of $(G,\cdot)$ onto $(G,\circ)$.

An autotopism is an isotopism of a quasigroup $(G,\cdot)$ onto itself.Although the component of an isotopism are usually denoted by lower case Greek letters we shall dehnote the components of an autotopism by capital letters. If $T = (U, V, W)$ is an autotopism of a quasigroup $(G,\cdot)$ then we have $(x)U \cdot (y)V = (x \cdot y)W$ for all $x, y \in G$.

The properties of autotopism of quasigroups and loops is widely discuss in [,] but let recall that:

Definition:4 A bijection U of a quasigroup $(Q,\cdot)$ is called autotopic, if there exists an autotopism(U, V, W) of $(Q,\cdot)$.

The fact that autotopism of a quasigroup form a group with identity (i, i, i) and that multiplication of autotopisms is componentwise allows us to conclude that autotopic bijection of $(Q,\cdot)$ form a group which we will denote by $\sum$.

Definition:5 A bijection U of a quasigroup $(Q,\cdot)$ is called λ −regular if there exists an autotopism (U, i, U) of a quasigroup $(Q,\cdot)$. A bijection U is called ρ −regular if there exists a bijection (i, U, U) of $(Q,\cdot)$.

We can rephrase this definition using identical relation. U is λ −regular if and only if $(x)U \cdot y = (x \cdot y)U$ for all $x, y \in Q$

Similarly, U is ρ −regular if and only if $x \cdot (y)U = (x \cdot y)U$ for all $x, y \in Q$.

Definition:6 A bijection U of a quasigroup $(Q,\cdot)$ is called μ −regular if there exists another bijection U' called the adjoint of U, such that $(x)U \cdot y = x \cdot (y)U'$ for all $x, y \in Q$.

Definition:7 A group Π of bijections on a non empty set S is called transitive on S, if any element $a_i \in S$ can be mapped on any other element $a_j \in S$ by some bijection $\pi \in \Pi$, that is

$$a_j = (a_i)\pi \text{ for some } \pi \in \Pi.$$

Definition:8 A quasigroup $(Q,\cdot)$ is called Λ −transitive (P −transitive, Θ −transitive) if the group $\Lambda(\mathrm{P}, \Theta)$ is transitive on the set Q.

Since every entropic quasigroup $(Q,\cdot)$ is Θ −transitive, every loop isotope of $(Q,\cdot)$ is a group.In fact every loop isotope of an entropic quasigroup is an abelian group. This fact accounts for the name”abelian” used by Murdoch.

A construction method for entropic quasigroup with one sided identity is given by the following result:

Theorem:1 If $(Q,\cdot)$ is an entropic quasigroup and if g is a fixed element of Q, then the principal isotope $(Q,\circ)$ such that $x \circ y = xR(g)^{-1} \cdot y$ is an entropic quasigroup with right identity element $e_\rho = g$. The principal isotope $(Q,*)$ such that $x * y = x \cdot yL(g)^{-1}$ is an entropic quasigroup with left identity $e_\lambda = g$.

Proof: We will start our proof with the case $(Q,\circ)$ such that $x \circ y = xR(g)^{-1} \cdot y = (x/g) \cdot y$ for all $x, y \in Q$. Then g is the right identity $x \circ g = (x/g) \cdot g = x$. To prove that $(Q,\circ)$ is entropic, let $s \in Q$ be the local left identity element for g in $(Q,\cdot)$, $s \cdot g = g$. Then we have $(x \cdot y) \cdot g = (x \cdot y) \cdot (s \cdot g) = (x \cdot s) \cdot (y \cdot g)$. This means that $A = \big(R(s), R(g), R(g)\big)$ is an autotopism of $(Q,\cdot)$. Applying A to the product $xR(s)^{-1} \cdot yR(g)^{-1}$ we have that

$$(xR(s)^{-1}, yR(g)^{-1})R(g) = x \cdot y \qquad (v.1)$$

We now want to show that:

$$(x \circ u) \circ (v \circ y) = (x \circ v) \circ (u \circ y)$$

Using the definition of $(Q,\circ)$ and $(v.1)$, we then have

$$\begin{aligned}(x \circ u) \circ (v \circ y) &= [(xR(g)^{-1} \cdot u)R(g)^{-1})]\,(uR(g)^{-1} \cdot y)\\ &= [xR(g)^{-1}R(s)^{-1} \cdot uR(g)^{-1})R(g)R(g)^{-1}](vR(g)^{-1} \cdot y)\\ &= [(xR(g)^{-1}R(s)^{-1}) \cdot uR(g)^{-1}](vR(g)^{-1} \cdot y)\\ &= (xR(g)^{-1}R(s)^{-1} \cdot vR(g)^{-1})(uR(g)^{-1} \cdot y\\ &= [(xR(g)^{-1} \cdot v)R(g)^{-1}][uR(g)^{-1} \cdot y] = (x \circ v) \circ (u \circ y)\end{aligned}$$

Hence $(Q,\circ)$ is entropic.

For the case $(Q,*)$, we have for all $x, y \in Q$, $x * y = x \cdot yL(g)^{-1} = x \cdot (g\backslash y)$. Then g is the left identity: $g * x = g \cdot (g \setminus x) = x$. To prove that $(Q,*)$ is entropic let $r \in Q$,be the local right identity element for g in $(Q,*)$: $g \cdot r = g$. Then we have the following $g \cdot (x \cdot y) = (g \cdot r) \cdot (x \cdot y) = (g \cdot x) \cdot (r \cdot y)$. This means that $B = \big(L(g), L(r), L(g)\big)$ is an autotopism of $(Q,\cdot)$. Applying B to the product $xL(g)^{-1} \cdot yL(r)^{-1}$,we have

$$(xL(g)^{-1}.yL(r)^{-1})L(g) = (xL(g)^{-1})L(g) \cdot (yL(g)^{-1})L(r) = x \cdot y \qquad (v.2)$$

We want to show that

$$(x * u) * (v * y) = (x * v) * (u * y)$$

Using the definition of $(Q,*)$ and $(v.2)$, we then have

$$\begin{aligned}(x * u) * (v * y) &= (x \cdot uL(g)^{-1}) * (v \cdot yL(g)^{-1})\\ &= (x \cdot uL(g)^{-1}) \cdot (v \cdot yL(g)^{-1})L(g)^{-1}\\ &= (x \cdot uL(g)^{-1}) \cdot (vL(g)^{-1} \cdot yL(g)^{-1}L(r)^{-1}\\ &= (x \cdot vL(g)^{-1}) \cdot (uL(g)^{-1} \cdot yL(g)^{-1}L(r)^{-1})\\ &= (x \cdot vL(g)^{-1}) \cdot [u \cdot yL(g)^{-1}]L(g)^{-1} = (x * v) * (u * y).\end{aligned}$$

Theorem:2 The parastrophs of an entropic quasigroup are entropics.

Proof: In literature,the six parastrophs of any quasigroup are well known we want just to prove the claim that whenever a quasigroup$(Q,\cdot) = (Q, F)$ satisfy the entropic identity, the following algebraic structures(Q, F^{-1}); $(Q,{}^{-1}F)$;$(Q,{}^{-1}(F^{-1}))$;$(Q,({}^{-1}F)^{-1})$ and $(Q,({}^{-1}(F^{-1}))^{-1})$ called parastrophs of (Q, F) are also entropic quasigroups.

1-) For the case of (Q, F) it is obvious.

2-) For the case (Q, F^{-1}) we are aware that $F^{-1}(c, b) = c/b = a \leftrightarrow a \cdot b = c, \forall a, b, c \in Q$. We want to show that for all a, u, v, b in Q,

$$(a/u)/(v/b) = (a/v)/(u/b). \qquad \textbf{(1)}$$

Let $x = a/u$; $y = v/b$; $z = a/v$; $t = u/b$ and $w = x/y$, this imply that : $a = x \cdot u$, $v = y \cdot b$; $a = z \cdot v$; $u = t \cdot b$ and $x = w \cdot y$, equation (1) is reduces to $x/y = z/t$ or to $w = z/t$ so to solve relation (1) one just need to prove that $z = w \cdot t$.

But $w \cdot t = (x/y) \cdot (u/b) = (x \cdot u)/(y \cdot b) = a/v = z$.

Hence (Q,/) is entropic

3-) For the case $(Q,{}^{-1}F)$ we are aware that ${}^{-1}F(a, c) = a\backslash c = b \leftrightarrow a \cdot b = c, \forall a, b, c \in Q$. We want to show that for all a, u, v, b in Q,

$$(a\backslash u)\backslash(v\backslash b) = (a\backslash v)\backslash(u\backslash b). \qquad \textbf{(2)}$$

Let take $i = a\backslash u$; $j = v\backslash b$; $k = a\backslash v$; $l = u\backslash b$, and $p = i\backslash j$. We then have the following relation $u = a \cdot i$; $b = v \cdot j$; $v = a \cdot k$; $b = u \cdot l$ and $j = i \cdot p$. So the relation (2) become $i\backslash j = k\backslash l$ or $p = k\backslash l$ to solve (2), one only need to prove that $l = k \cdot p$.

But $k \cdot p = (a\backslash v) \cdot (i\backslash j) = (a \cdot i)\backslash(v \cdot j) = u\backslash b = l$.

Hence the quasigroup (Q,\) is entropic

4-) For the case $(Q,({}^{-1}(F^{-1}))^{-1})$ we are aware that $({}^{-1}(F^{-1}))^{-1}(b, a) = b \circ a = c$ if and only if $a \cdot b = c$ for all $a, b, c \in Q$. We want to show that for all a, u, v, b in Q,

$$(a \circ u) \circ (v \circ b) = (a \circ v) \circ (u \circ b). \qquad \textbf{(3)}$$

But
$$\begin{aligned}(a \circ u) \circ (v \circ b) = (u \cdot a) \circ (b \cdot v) &= (b \cdot v) \cdot (u \cdot a)\\ &= (b \cdot u) \cdot (v \cdot a)\\ &= (u \circ b) \cdot (a \circ v)\\ &= (a \circ v) \circ (u \circ b)\end{aligned}$$

This prove that $(Q,\circ)$ is entropic.

5-) For the case $(Q,{}^{-1}(F^{-1}))$ we are aware that $({}^{-1}(F^{-1}))(c,a) = c * a = b$ if and only if we have $a\backslash c = b$ for all $a,b,c \in Q$. We want to show that for all a,u,v,b in Q,

$$(a * u) * (v * b) = (a * v) * (u * b) \quad \textbf{(4)}$$

But we know that $(a * u) * (v * b) = (u\backslash a) * (b\backslash v) = (b\backslash v)\backslash(u\backslash a)$

$$=(b\backslash u)\backslash(v\backslash a)$$

$$= (u * b)\backslash(a * v)$$

$$= (a * v) * (u * b)$$

Hence the parastroph $(Q,*)$ of $((Q,\cdot)$ is entropic.

6-) For the case $(Q,({}^{-1}F)^{-1})$ we are aware that $(({}^{-1}F)^{-1})(b,c) = b \times c = a \leftrightarrow c/b = a$,for all $a,b,c \in Q$. We want to show that for all a,u,v,b in Q

$$(a \times u) \times (v \times b) = (a \times v) \times (u \times b(5)$$

But we have that: $(a \times u) \times (v \times b) = (u/a) \times (b/v) = (b/v)/(u/a)$

$$=(b/u)/(v/a)$$

$$= (u \times b)/(a \times v)$$

$$= (a \times v) \times (u \times b)$$

Hence $(Q,\times)$ is also entropic, therefore the theorem is proved.

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