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Solution of Nonlinear Brusselator Model by a Combined Sawi Transform and Homtopy Analysis Method

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Abstract

The Brusselator is a theoretical model for a type of autocatalytic reaction to analyze the behaviour of the chemical systems with non-linear oscillator. Fractional-order Brusselator system of equations (Reaction-Diffusion system) were solved using Sawi Homotopy Analysis Method (SHAM) which is a combination of Sawi transform and Homotopy analysis method. Obtained results were compared with the results in the literature and it was deduced that the mean absolute error (MAE) obtained by SHAM were smaller compared to the solution in the literature. The compiled findings showed the efficacy of the implemented technique and hence recommended for solving fractional-order nonlinear partial differential equations.(within the domains of applied sciences, engineering and technology).

Keywords: Brusselator system, Sawi integral transform, Homotopy analysis method, Noninteger differential equations.

1. INTRODUCTION

Nonlinear fractional-order partial differential equations are of significant importance in the field of science , engineering and technology. However solving these set of nonlinear equations poses a chalenge due to the absence of a precise approach. Numerous numerical methods such Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Differential Transform Method (DTM), Homotopy Analysis Method (HAM), Homotopy Perturbation Method(HPM) have been extensively employed to solve different kinds of differential equations such as Ordinary Differential Equations (ODEs), Partial Differential Equations (PDEs) as well as Integro-Differential Equations, and the methods were discovered to be efficient, effective with wide applicability [4, 6, 7,9, 11,18,25,27]. Several authors have applied HAM for solving fractional barrier option PDE, systems of ODE, and integro-differential equations [4, 6,26,30]. Several integral transformations such as Laplace, Elzaki, Aboodh, Sumudu, Mohand, Sawi, Mahgoub, Kamal, Sadik have been used extensively for solving linear differential equations, be it classical or fractional order differential equations, arising from mathematical physics, applied mathematics, engineering sciences, signal processing, which are defined by differential equations, by converting complex equations into an algebraic form that can be easily simplified through the use of integration [9,10,15,18,19,20, 21,22,27].

The benefits of using integral transforms include simplifying and compactly representing the function, as well as doing frequency analysis by breaking the function down into its frequency components. Usually, they employ mathematical tools to find solutions to cutting-edge issues in engineering, science, technology, and space exploration.

Several numerical techniques such as Homotopy Pertubation Method (HPM), Homotopy Analysis Method (HAM), Variational Iteration Method (VIM), Adomian Decomposition Method (ADM), least square method, residual power series method were combined with various integral transforms such as Sawi, Mohand, Laplace, Elzaki, Shehu for finding the approximate solution of reaction-difussion system of classical type and fractional order differential equations [4,5,6,7,9,11,18,19,20,21,24,28,29,30]

A mathematical model called the Brusselator is used to describe the behaviour of certain chemical reactions and pattern development. It as well describes how concentration changes over time as well as auto-catalytic processes in which one of the reactants acts as a catalyst for its own synthesis [1, 2, 3, 5, 7, 8,12,13,14,16,17,23]. In this work, HAM is incorporated into the scheme of Sawi transform for solving nonlinear Brusselator system and the obtained solution is in the form of super-convergent series. Two applications where considered in which the results obtained were compared with the exact solution as well as the reference solutions.

Concept of Homotopy Analysis Method

The basic idea of HAM is explained in this section by considering the differential equation

$$N[u(x,t)] = 0, \quad (x,t) \in \Omega \tag{1}$$

where N is the operator both linear and nonlinear, x and t are the independent variables and u is the unknown function in the domain Ω.

The homotopy operator H is defined as [6,9]

$$H(\phi, s) \equiv (1-s)[L(\phi(x;s)) - u_0(x)] - shN(\phi(x;s)) \tag{2}$$

where $s \in [0,1]$ is an embedding parameter and $h \neq 0$ is the convergence control parameter, u_0 is an initial guess of the solution of Eqn. (1), ϕ is an unknown function and L is the auxiliary linear operator satisfying the feature $L(0) = 0$. When $H(\phi, s)$ is consider to be zero, gives

$$(1-s)[L(\phi(x;t;s)) - u_0(x,t)] = shN(\phi(x;t;s)) \tag{3}$$

which is known as the zero-order deformation equation. From Eqn. (3), it can be observed that

for,

$$s = 0, \quad L(\phi(x,t;0)) - u_0(x,t) = 0$$

which gives $\phi(x;t;0) = u_0(x,t)$. Conversely, for $s = 1$, Eqn. (3) reduces to $N(\phi(x;t;s)) = 0$ and this gives $\phi(x;t;1) = u(x)$. Thus, by replacing s from 0 to 1, the result changes from u_0 to u.

Using Maclaurin Series, the function $\phi(x;t;s)$ with parameter s may be written as

$$\phi(x;t;s) = \phi(x,t;0) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k \phi(x;t;s)}{\partial s^k}\bigg|_{s=0} s^k \tag{4}$$

representing

$$y_k(x,t) = \frac{1}{k!} \frac{\partial^k (x;t;s)}{\partial s^k}\bigg|_{s=0} k = 1,2,3 \tag{5}$$

Eqn. (4) changes to

$$\phi(x;t;s) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t) s^k \tag{6}$$

If the series (6) converged for $s = 1$, the solution of Eqn. (1) is given as

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) \tag{7}$$

To find the function u_k, Eqn. (3) is differentiated, k times with respect to s and the result is divided by $k!$ where $s = 0$ through this kth-order deformation equation for $k > 0$ is defined thus

$$L(y_k(x,t) - x - ky_{k-1}(x,t)) = hH(x,t)\phi_k(y_{k-1}(x,t)) \tag{8}$$

where $H(x,t)$ is the auxiliary function,

$$X_n = \begin{cases} 0, & n \leq 1 \\ 1, & n > 1 \end{cases} \quad \text{and} \quad \phi k(y_{k-1}(x,t)) = \frac{1}{(k-1)!}\left(\frac{\partial^{k-1}}{\partial p^{k-1}} N\left(\sum_{j=1}^{\infty} y_j(x)p^j\right)\right)$$

2. DEFINITION OF SAWI TRANSFORM

Sawi transform of the function $f(t)$ for all $t \geq 0$ is defined as:

$$S(f(t)) = \frac{1}{w^2}\int_0^{\infty} f(t)e^{-(\frac{t}{w})}dt = f(w), w > 0, \tag{9}$$

where S stands for Sawi transform operator [14]

2.1. Some fundamental properties of Sawi transform [24]

S/n	f(t)	S(f(t))=F(w)
1	1	$\frac{1}{w}$
2	t	1
3	t^2	2!w
4	$t^k, k \in K$	kw^{k-1}
5	$t^k, k > -1$	$\Gamma(k+1)w^{k-1}$
6	e^{xt}	$\frac{1}{w(1-xw)}$

There are some important theorems that will be useful in the derivation of schemes needed to solve the Brusselator models, which are stated as follows.

2.2. Theorem 1:

Let $F(w)$ be the Sawi transform of $f(t)$ where $S(f(t)) = F(w)$ then the Sawi transform of nth order derivatives is given as

$$S\Big(f^n(t)\Big) = \frac{1}{w^n}F(w) - \sum_{k=1}^{n} \frac{f^{k-1}(0)}{w^{n+2-k}} \tag{10}$$

Proof: The principle of Mathematical induction is employed to establish Eqn.(10). The procedure involves showing that Eqn. (10) is true for $n = 1$, then assuming that is true for any natural number $n = m$ and then show that it is true for $n = m + 1$ whenever it is true for $n = m$

When $n = 1$, then Eqn. (10) gives

$$S\Big(f'(t)\Big) = \frac{1}{w}F(w) - \frac{1}{w^2}f(0) \tag{11}$$

From the definition of Sawi's transform,

$$S\Big(f'(t)\Big) = \frac{1}{w^2}\int_0^{\infty} e^{-(\frac{t}{w})}f'(t)dt \tag{12}$$

Simplifying the RHS of the Eqn. (12) using integration by parts,

$$S\Big(f'(t)\Big) = -\frac{1}{w^2}f(0) + \frac{1}{w}\cdot\frac{1}{w^2}\int_0^{\infty} e^{-(\frac{t}{w})}f(t)dt \tag{13}$$

where $F(w) = \frac{1}{w^2}\int_0^{\infty} e^{-\frac{t}{w}}f(t)dt$

Hence

$$S\Big(f'(t)\Big) = \frac{1}{w}R(w) - \frac{1}{w^2}f(0) \tag{14}$$

This established that $n = 1$ is true in Eqn.(10)

Assuming $n = m$ is true in Eqn. (10) then

$$S\Big(f^m(t)\Big) = \frac{1}{w}F(w) - \sum_{k=1}^{m} \frac{f^{k-1}(0)}{w^{m+2-k}} \tag{15}$$

$$S\Big(f^m(t)\Big) = \frac{1}{w^2}\int_0^{\infty} e^{-(\frac{t}{w})}f^m(t)dt \tag{16}$$

where $f^m(t)$ is the mth derivative of $f(t)$

It has to be shown that $n = m + 1$ is true in Eqn. (10) whenever $n = m$ is true.

When $n = m + 1$, Eqn. (10) becomes

$$S\Big(f^{m+1}(t)\Big) = \frac{1}{w^{m+1}}F(w) - \sum_{k=1}^{m+1} \frac{f^{k-1}(0)}{w^{m+3-k}} \tag{17}$$

To establish Eqn. (16), from the definition of Sawi's transform,

$$S\Big(f^{m+1}(t)\Big) = \frac{1}{w^2}\int_0^\infty e^{-(\frac{t}{w})} f^{m+1}(t)dt, \tag{18}$$

The RHS of Eqn. (18), is simplified using integration by parts

$$S\Big(f^{m+1}(t)\Big) = \frac{1}{w^2}\Big(-f^m(0)\Big) + \frac{1}{w}\Big(\frac{1}{w^m}F(w) - \sum_{k=1}^{m}\frac{f^{k-1}(0)}{w^{m+2-k}}\Big) \tag{19}$$

Further simplification of Eqn. (19), yields

$$S\Big(f^{m+1}(t)\Big) = -\frac{1}{w^2}f^m(0) + \Big(\frac{1}{w^{m+1}}F(w) - \sum_{k=1}^{m}\frac{f^{k-1}(0)}{w^{m+3-k}}\Big) \tag{20}$$

Since,

$$\frac{1}{w^2}\int_0^\infty e^{-(\frac{t}{w})} f^m(t)dt = \frac{1}{w^m}F(w) - \sum_{k=1}^{m}\frac{f^{k-1}(0)}{w^{m+2-k}} \tag{21}$$

The essence of this Theorem 1 is to solve fractional differential equations using Sawi derivative properties.

2.3. Theorem 2:

If f(t) is a piecewise continuous function of an exponential order and Caputo fractional derivative is given then, the Sawi transform of Caputo fractional order derivative is given as:

$$S({}_0^cD_x^\alpha f(x)) = w^{-\alpha}p(w) - w^2\sum_{m}^{2} = 1^n w^{m-\alpha} f^{m-1}(0) \tag{22}$$

Proof:
According to the definition of Caputo fractional derivative

$$D^\alpha f(x) = \frac{1}{\Gamma n-\alpha}\int_0^\infty (x-t)^{n-\alpha-1f(t)}dt \tag{23}$$

Taking the Sawi transform of Eqn. (21) gives:

$$S({}_0^cD_x^\alpha f(x)) = S({}_0I_x^{n-\alpha} f^n(x)) \tag{24}$$

It is necessary to show that

$$\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1} f(t)dt = I^{n-\alpha} f^n(x) \tag{25}$$

and by convolution theorem as proved by [29]

$$f * g = \int_0^t f(t-x)g(x)dx \tag{26}$$

and so,

$$\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha} f(t)dt = \frac{1}{\Gamma(n-\alpha)}(x^{n-\alpha} * f(x)), \tag{27}$$

Eqn. (27) becomes

$$I^{n-\alpha} f^n(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x^{n-\alpha} * f(x)) \tag{28}$$

applying the definition of Sawi's convolution theorem on Eqn. (28) yields

$$S(I^{n-\alpha} f^n(x)) = \frac{1}{\Gamma(n-\alpha)} * \Gamma(n-\alpha) w^{n-\alpha} \tag{29}$$

Therefore,

$$S(I^{n-\alpha} f^n(x)) = w^{n-\alpha} g(w) \tag{30}$$

where $G(w)$ is $S(g(x))$ and $S(g(x))$ is $S(f^n(x))$, therefore

$$S\Big(g(x)\Big) = \frac{F(w)}{w^n} - w^2 \sum_{m=1}^{n} w^{m-n} F^{m-1}(0) \tag{31}$$

putting Eqn. (31) in Eqn. (30) gives;

$$S(I^{n-\alpha} f^n(x)) = w^{n-\alpha}\Big[\frac{F(w)}{w^n} - w^2 \sum_{m=1}^{n} w^{m-n} F^{m-1}(0)\Big] \tag{32}$$

Simplifying Eqn. (32) gives;

$$S({}_0^c D_x^\alpha f(x)) = w^{-\alpha} F(w) - w^2 \sum_{m=1}^{n} w^{m-\alpha} f^{m-1}(0) \tag{33}$$

The importance of this theorem is to find Sawi transform of Caputo derivative of a function which has order n and this derivative will be used to formulate the scheme for solving nonlinear fractional order differential equation of Caputo type.

2.4. The Scheme of Sawi transform and Homotopy Analysis Method

Consider the non-linear fractional differential equation

$$D^{\alpha}(y(x,t)) + R(y(x,t)) + F(y(x,t)) = g(x,t), \quad j-1 < \alpha \leq j \tag{34}$$

where $D^{\alpha}(y(x,t))$ is the Caputo derivative of order α, $R(y(x,t))$ is the linear operator with fractional order less than α, F is the nonlinear operator and $g(x,t)$ is the nonhomogeneous term.
Applying Sawi transform (S) on both sides of Eqn. (34), gives

$$S\Big(D_t^{\alpha}y(x,t) + Ry(x,t) + Fy(x,t)\Big) = S[g(x,t)] \tag{35}$$

where

$$S(D_t^{\alpha}y) = \Big[\frac{Y(w)}{w^{\alpha}} - \sum_{k=0}^{n-1}\frac{y^k(0)}{w^{\alpha-k+1}}\Big] \tag{36}$$

substituting Eqn. (36) into Eqn. (35) gives

$$\Big(\frac{Y(w)}{w^{\alpha}} - \sum_{k=0}^{n-1}\frac{y^k(0)}{w^{\alpha-k+1}}\Big) + S(Ry) + S(Fy) = S(g(x,t)) \tag{37}$$

or

$$\bar{y}(x,w) - \sum_{k=0}^{n-1}\frac{y^k(0)}{w^{-k+1}} + w^{\alpha}S(R(y)) + w^{\alpha}S(F(y)) - w^{\alpha}S(g(x,t)) = 0 \tag{38}$$

By applying nth order-deformation equation on Eqn. (38), gives

$$S(\bar{y}_n(x,w) - \chi_n\bar{y}_{n-1}) =$$

$$hD_{n-1}\Big(\bar{\phi}(x,w,q) - \sum_{k=0}^{n-1}\frac{y^k(0)}{w^{-k+1}} + w^{\alpha}S(R(y)) + w^{\alpha}S(F(y)) - w^{\alpha}S(g(x,t))\Big) \tag{39}$$

putting $h=-1$ and using homotopy derivative property on Eqn. (39), leads to

$$\bar{y}_n(x,w) = \chi_n\bar{y}_{n-1}$$

$$-\Big(\bar{y}_{n-1}(x,w) + (1-\bar{\chi}_{n-1})\Big(-\sum_{k=0}^{n-1}\frac{y^k(0)}{w^{-k+1}} - w^{\alpha}S[g(x,t)] + D_{n-1}(w^{\alpha}S(R(y)+F(y))\Big)\Big)\Big) \tag{40}$$

simplifying Eqn. (40), gives

$$\bar{y}_n = -(1-\chi_n)\bar{y}_{n+1} - (1-\bar{\chi}_{n-1})\Big(\sum_{k=0}^{n-1}\frac{y^k(0)}{w^{-k+1}} + S(g(x,t))\Big) - w^\alpha S\Big(R(y)+F(y)\Big) \tag{41}$$

$$y_0(x,t) = S^{-1}\Big(-(1-\bar{\chi}_{n-1})\Big(\sum_{k=0}^{n-1}\frac{Y^k(0)}{w^{-k+1}} + S(g(x,t))\Big)\Big) \tag{42}$$

The nonlinear part of Eqn. (41) is decomposed thus

$$D_{n-1}\Big(w^\alpha S\Big(\phi(x,w,q)\frac{\partial\phi(x,w,q)}{\partial x}\Big)\Big) = w^\alpha S\Big(\sum_{i=0}^{n-1} y_i\frac{\partial y_{n-i-1}}{\partial x}\Big) \tag{43}$$

or

$$D_{n-1}\Big(w^\alpha S\Big(\phi^2(x,w,q)\Big)\Big) = w^\alpha S\Big(\sum_{i=0}^{n-1} y_i y_{n-1-i}\Big) \tag{44}$$

where

$$\chi_n = \begin{cases} 0, & n \le 1 \\ 1, & n > 1 \end{cases} \quad and \quad \bar{\chi}_{n-1} = \begin{cases} 0, & n-1 < 1 \\ 1, & n-1 \ge 1 \end{cases} \tag{45}$$

Taking the Sawi inverse of (41) using Eqn. (45) leads to Eqn. (46)

$$y_n(x,t) = y_0(x,t) + S^{-1}\Big[w^\alpha S\Big[R(y_n)+F(y_n)\Big]\Big] \tag{46}$$

3. NUMERICAL APPLICATION I

(I) Consider the nonlinear system of Caputo-fractional Brusselator Model

$$\left.\begin{aligned} {}^cD_t^\psi p(x,y,t) - p^2(x,y,t)q + 2p(x,y,t) - \tfrac{1}{4}\Big(\tfrac{\partial^2 p}{\partial x^2}+\tfrac{\partial^2 p}{\partial y^2}\Big) = 0 \\ {}^cD_t^\psi q(x,y,t) - p(x,y,t) + p^2(x,y,t)q - \tfrac{1}{4}\Big(\tfrac{\partial^2 q}{\partial x^2}+\tfrac{\partial^2 q}{\partial y^2}\Big) = 0 \end{aligned}\right\} \tag{47}$$

Subject to the initial conditions

$$\left.\begin{aligned} p(x,y,0) &= e^{-x-y} \\ q(x,y,0) &= e^{x+y} \end{aligned}\right\} \tag{48}$$

The exact solution of Eqn. (51) is

$$\left.\begin{aligned} p(x,y,t) &= e^{-x-y-\frac{t}{2}} \\ q(x,y,t) &= e^{x+y+\frac{t}{2}} \end{aligned}\right\} \tag{49}$$

Taking Sawi transform of Eqn. (51) using Eqn. (52),yield

$$\left.\begin{aligned}\bar{P}(x,w)&= \chi_n\bar{p}_{n-1}(x,w)+hD_{n-1}\Big(\phi(x,w)-w^{-1}e^{-x-y}-w^{\psi}s\Big(p^2q-2p+\tfrac{1}{4}\Big(\tfrac{\partial^2p}{\partial x^2}+\tfrac{\partial^2p}{\partial y^2}\Big)\Big)\Big)\\ \bar{Q}(x,w)&=\chi_n\bar{q}_{n-1}(x,w)+hD_{n-1}\Big(\phi(x,w)-w^{-1}e^{x+y}-w^{\psi}s\Big(p-p^2q+\tfrac{1}{4}\Big(\tfrac{\partial^2p}{\partial x^2}+\tfrac{\partial^2p}{\partial y^2}\Big)\Big)\Big)\end{aligned}\right\} \tag{50}$$

Using the formulated scheme of Eqn. (50) in Eqn. (53) the initial approximation are obtained thus

$$\left.\begin{aligned}p_0(x,y,t)&= e^{-x-y}\\ q_0(x,y,t)&=e^{x+y}\end{aligned}\right\} \tag{51}$$

The recursive relations of other iterations were obtained as

$$\left.\begin{aligned}p_1(x,t)&= S^{-1}\Big(w^{\psi}s\Big(p_0^2q_0-2p_0+\tfrac{1}{4}\Big(\tfrac{\partial^2p_0}{\partial x^2}+\tfrac{\partial^2p_0}{\partial y^2}\Big)\Big)\Big)\\ q_1(x,t)&=S^{-1}\Big(w^{\psi}s\Big(p_0-p_0^2q_0+\tfrac{1}{4}\Big(\tfrac{\partial^2q_0}{\partial x^2}+\tfrac{\partial^2q_0}{\partial y^2}\Big)\Big)\Big)\end{aligned}\right\} \tag{52}$$

$$\left.\begin{aligned}p_1(x,t)&= \tfrac{-t^{\psi}}{2\Gamma(\psi+1)}e^{-x-y}\\ q_1(x,t)&=\tfrac{t^{\psi}}{2\Gamma(\psi+1)}e^{x+y}\end{aligned}\right\} \tag{53}$$

By using the decomposition formula, other iterations are

$$\left.\begin{aligned}p_2(x,t)&= \tfrac{t^{2\psi}e^{-x-y}}{2\Gamma(2\psi+1)}\\ q_2(x,t)&= \tfrac{t^{2\psi}e^{x+y}}{4\Gamma(2\psi+1)}\end{aligned}\right\} \tag{54}$$

Then, the series solution of Eqn. (51) is given as

$$\left.\begin{aligned}p(x,y,t)&= \textstyle\sum_{n=0}^{\infty}p_n(x,y,t)\\ q(x,y,t)&= \textstyle\sum_{n=0}^{\infty}q_n(x,y,t)\end{aligned}\right\} \tag{55}$$

Numerical Application 2

$$\left.\begin{aligned}\tfrac{\partial^{\psi}p}{\partial t}-1-p^2q+\tfrac{3}{2}p-\tfrac{1}{4}\Big[\tfrac{\partial^2p}{\partial x^2}+\tfrac{\partial^2p}{\partial y^2}\Big]&=0\\ \tfrac{\partial^{\psi}q}{\partial t}-\tfrac{1}{2}p+p^2q-\tfrac{1}{4}\Big[\tfrac{\partial^2q}{\partial x^2}+\tfrac{\partial^2q}{\partial y^2}\Big]&=0\end{aligned}\right\} \tag{56}$$

Subject to initial conditions

$$p(x,y,0)=x^2 \qquad and \qquad q(x,y,0)=y^2 \tag{57}$$

Using the Scheme formulated in Eqn. (50) together with the given initial conditions on Eqn. (56) leads to

$$\left.\begin{aligned} p_1(x,t) &= p_0 + s^{-1}\Big(w^{\psi}s\Big(1 + p_0^2q_0 - \tfrac{3}{2}p_0 + \tfrac{1}{4}\Big(\tfrac{\partial^2 p_0}{\partial x^2} + \tfrac{\partial^2 p_0}{\partial y^2}\Big)\Big)\Big) \\ q_1(x,t) &= q_0 + s^{-1}\Big(w^{\psi}s\Big(\tfrac{1}{2}p_0 + p_0^2q_0 + \tfrac{1}{4}\Big(\tfrac{\partial^2 q_0}{\partial x^2} + \tfrac{\partial^2 q_0}{\partial y^2}\Big)\Big)\Big) \end{aligned}\right\} \tag{58}$$

$$p_1(x,t) = \frac{t^{\psi}}{\Gamma(\psi+1)}\Big(\frac{3}{2} - \frac{3}{2}x^2 + x^4y^2\Big) \quad and \quad q_1 = \frac{t^{\psi}}{\Gamma(\psi+1)}\Big(\frac{1}{2} + \frac{x^2}{2} + x^4y^2\Big) \tag{59}$$

By using the decomposition formula for the nonlinear terms in Eqn. (56) other iteration were obtained thus

$$\left.\begin{aligned} p_2(x,t) &= \tfrac{t^{2\psi}}{\Gamma(2\psi+1)}\Big(\tfrac{9}{4}x^2 - \tfrac{9}{2}x^4y^2 + 6x^2y^2 - \tfrac{1}{2}x^6 + x^4 + 2x^6y^4 - x^8y^6 - 3\Big) \\ q_2(x,t) &= \tfrac{t^{2\psi}}{\Gamma(2\psi+1)}\Big(1 - \tfrac{3}{4}x^2 + \tfrac{7}{2}x^4y^2 - \tfrac{1}{2}x^6 - 6x^2y^2 - x^4 - 2x^6y^4 + x^8y^6\Big) \end{aligned}\right\} \tag{60}$$

The series solution is given as

$$\left.\begin{aligned} p(x,y,t) &= \textstyle\sum_{n=0}^{2} p_n(x,y,t) \\ q(x,y,t) &= \textstyle\sum_{n=0}^{2} q_n(x,y,t) \end{aligned}\right\} \tag{61}$$

4. RESULTS

4.1. Table 1: Results of p(x,y,t) and q(x,y,t) for Application 1 at $x = 1$

t	AESHAM (p)	AELADM (p)[7]	AESHAM (q)	AELADM (q)
0.1	0.0000027847	0.0000056040	0.000001943	0.000007582
0.2	0.0000220106	0.0000445589	0.000031414	0.0000445589
0.3	0.0000734128	0.0001494809	0.000160658	0.000312910
0.4	0.0001720181	0.0003522209	0.000512983	0.000873877
0.5	0.0003322108	0.0006839010	0.001265375	0.01970246
0.6	0.0005677973	0.0011749511	0.002651249	0.003869267
0.7	0.00089206811	0.00185514134	0.004963340	0.006897510
0.8	0.00131785958	0.00275361562	0.008556750	0.011443900
0.9	0.00185761345	0.00389892128	0.013852160	0.017962970
1.0	0.0003284114	0.000402875	0.021339130	0.026978100
MAE	$7.76121098 \times 10^{-4}$	$1.62374335 \times 10^{-3}$	5.3335002×10^{-3}	7.0392888×10^{-3}

AESHAM: Absolute Error Of Sawi Homotopy Analysis Method
AELADM: Absolute Error Of Laplace Transform Adomian Decomposition Method

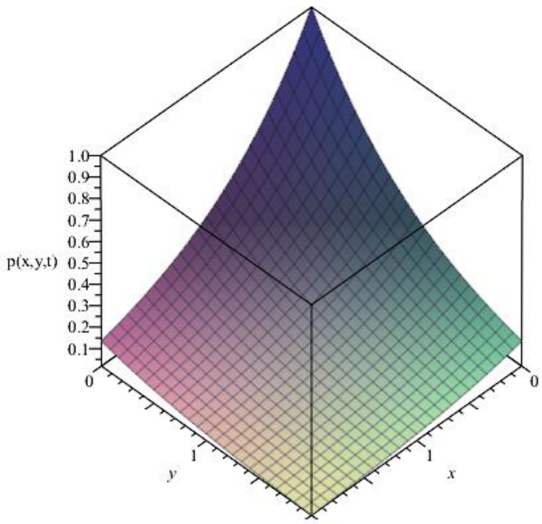

Figure 1: $p(x, y, t)$ at $t = 0$ and $\psi = 1$

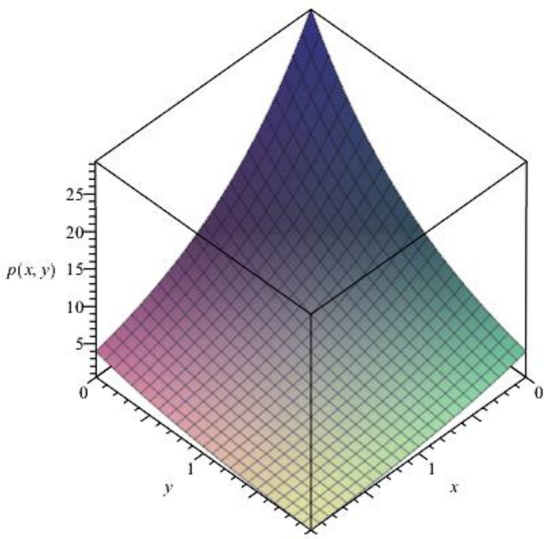

Figure 2: $p(x, y, t)$ at $t = 10$ and $\psi = 1$

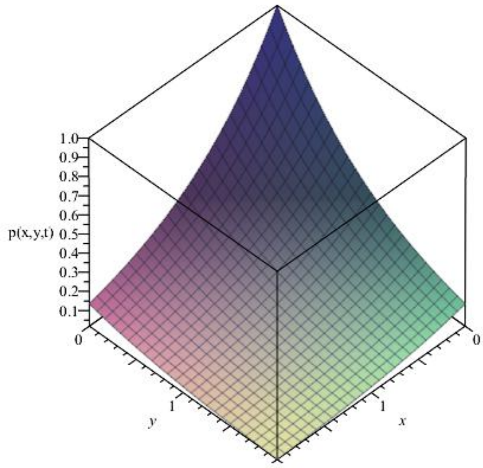

Figure 3: Exact of $p(x, y, t)$ at $t = 0$

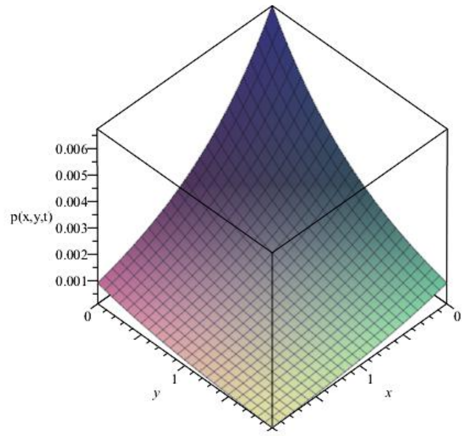

Figure 4: Exact of $p(x, y, t)$ at $t = 10$

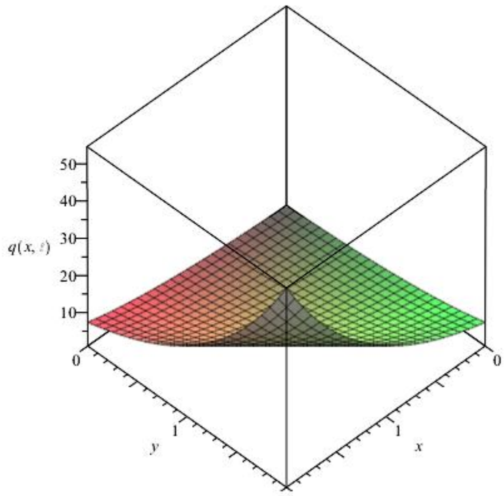

Figure 5: $q(x, y, t)$ at $t = 0$ and $\psi = 1$

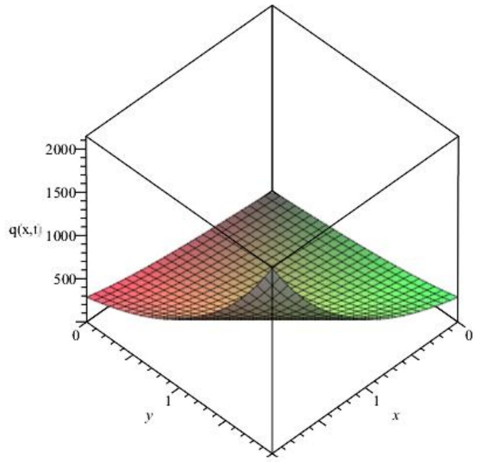

Figure 6: $q(x, y, t)$ at $t = 10$ and $\psi = 1$

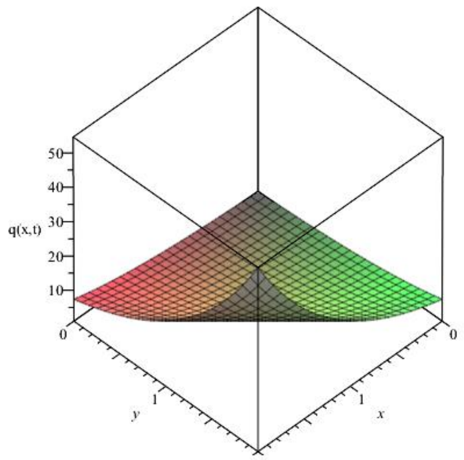

Figure 7: Exact of $q(x, y, t)$ at $t = 0$

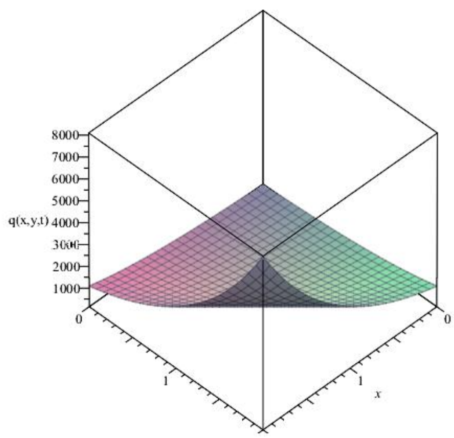

Figure 8: Exact of $q(x, y, t)$ at $t = 10$

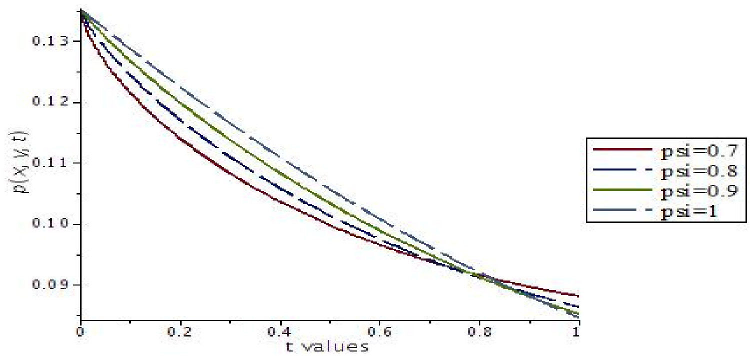

Figure 9: 2D plot of $p(x, y, t)$

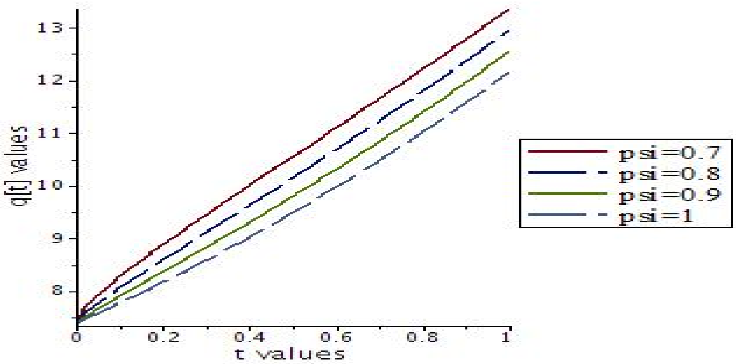

Figure 10: 2D plot of $q(x, y, t)$

4.2. Table 2: Result of P(x,y,t) and q(x,y,t) at $x = y = 1$ for Application 2

t	SHAM (p)	LADM (p)	SHAM (q)	LADM (q)
0.1	1.121250000	1.121250000	0.9762500000	0.9762500000
0.2	1.285000000	1.285000000	0.9050000000	0.9050000000
0.3	1.491250000	1.491250000	0.7862500000	0.7862500000
0.4	1.740000000	1.740000000	0.6200000000	0.6200000000
0.5	2.031250000	2.031250000	0.4062500000	0.4062500000
0.6	2.365000000	2.365000000	0.1450000000	0.1450000000
0.7	2.741250000	2.741250000	-0.163750000	-0.163750000
0.8	3.160000000	3.160000000	-0.5200000000	-0.5200000000
0.9	3.621250000	3.621250000	-0.923750000	-0.923750000
1.0	4.125000000	4.125000000	-1.375000000	-1.375000000

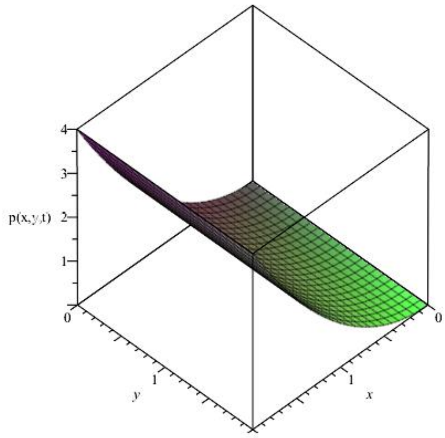

Figure 11: $p(x, y, t)$ at $t = 0$ and $\psi = 1$

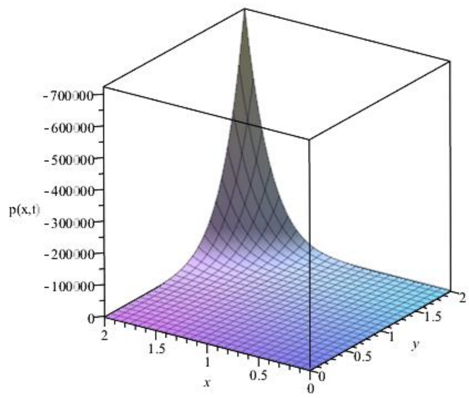

Figure 12: $p(x, y, t)$ at $t = 10$ and $\psi = 1$

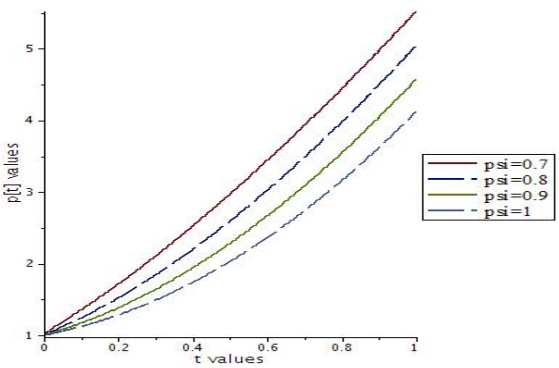

Figure 13: 2D plot of $p(x, y, t)$

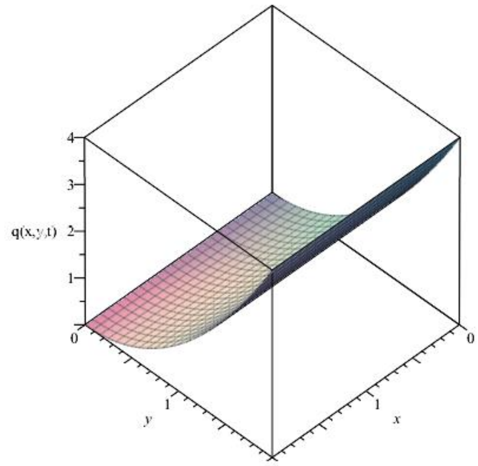

Figure 14: $q(x, y, t)$ at $t = 0$ and $\psi = 1$

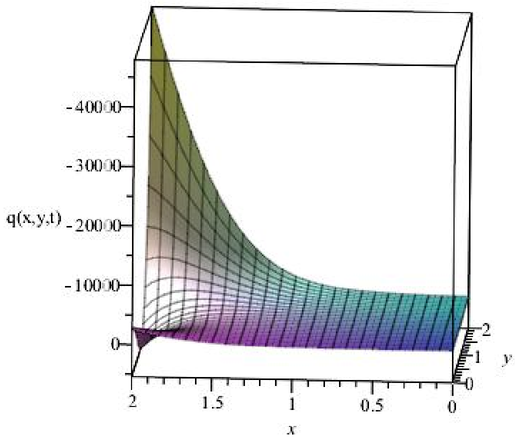

Figure 15: $q(x, y, t)$ at $t = 10$ and $\psi = 1$

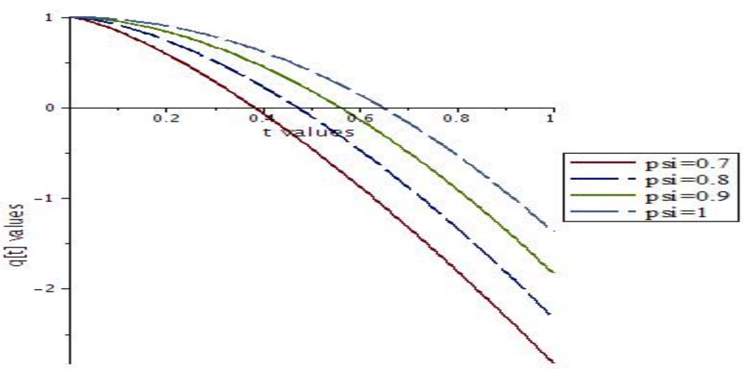

Figure 16: 2D plot of $q(x, y, t)$

5. DISCUSSION OF RESULTS

Sawi transforms of nth order derivative $(f^n(t))$ as well as Caputo derivative were established in Theorems 1 and 2. Sawi Homotopy Analysis Method (SHAM) was established and applied to solve nonlinear fractional-order Brusselator model. Two numerical applications were considered using the established scheme. The results were presented in form of Tables and Figures. The maximum and minimum errors for the two compartments of application 1 were 1.3179×10^{-3} and 2.7847×10^{-6} for $p(x, y, t)$ while it was 1.3852×10^{-2} and 1.9430×10^{-6} for $q(x, y, t)$, Mean Absolute Error (MAE) obtained for the first compartment $p(x, y, t)$ of application 1 was smaller as compared with that of [7] which is $7.776121098 \times 10^{-4}$ while the MAE for $q(x, y, t)$ is 5.3335002×10^{-3} which is slightly smaller as compared to the referenced error of $1.62374335 \times 10^{-3}$ and 7.0392888×10^{-3} for $p(x, y, t)$ and for $q(x, y, t)$ respectively from Table 1. This error was generated at the classical order $\psi = 1$, the graphical representation of the SHAM and exact of $p(x, y, t)$ and $q(x, y, t)$ of Eqn.(17) were presented in Figures 1-8 respectively at $\psi = 1$ for the values of $t = 0$ and $t = 10$ respectively. It was keenly observed from the 2D plots for the two compartments that, as the values of ψ increases from $0.7.0.8, 0.9$ and 1, the solution also increased and converged to the classical order for $p(x, y, t)$ while the solution decreased and converged towards classical order for $q(x, y, t)$. The graphs agrees with that of [7] and this established the effectiveness of the the technique employed. From Table 2, the same results were obtained for $P(x, y, t)$ and $q(x, y, t)$ as compared with the referenced solution and thus justify the effectiveness, reliability and applicability of the proposed method in solving non-linear Brusselator system. The solution of Eqn.(29) decreases and converged to the classical order for $p(x, y, t)$ while it was increased and coverged to the classical order for $q(x, y, t)$ as the values of ψ increases from $0.7.0.8, 0.9$ and 1 .

6. CONCLUSION

The scheme of SHAM was created as an improvement upon the integral transform proposed by Sawi with the main aim of solving nonlinear non-integer order Brusselator System. This work provided and demonstrated the validity of several pertinent theorems about Sawi transform of the Riemann-Liouville integral and Caputo derivative. In order to assess the efficacy as well as accuracy of SHAM technique, two applications were considered and the results performed well when compared with referenced solution.

REFERENCES

[1] Alfifi H.V., (2017), "Semi-Analytical Solutions for the Brusselator Reaction-Diffussion Model", *"ANZIAM*, 59(2017), pp.167-182.

[2] Alshammari S., Al-Sawalha M.M and Humaidi J.R.. (2023), Fractional View Study of the Brusselator Reaction-Diffusion Model occurring in Chemical Reactions, *Fractal Fract.*, 7(108), pp.1-12.

[3] Aminikhan H. and Jamalian A. (2013), An efficient method for solving the Brusselator system, *Walailak J. Sci & Tech* , 10(5), pp. 449-465.

[4] Bataineh A.S., Noorani, M.S.M. and Hashim I. (2008), Approximate analytical solutions of System of PDEs by Homotopy Analysis Method,*An International Journal of Computers and Mathematics with applications*, 55(2008), pp.2913-2923.

[5] Bota C and Caruntu B. (2015), Approximate Analytical Solution of the Fractional–order Brusselator System using the polynomial Least Squares Method,*Advances in Mathematical Physics*,Volume (2015), pp. 1-5.

[6] Fadugba S.E and Edeki S.O (2022), Homotopy Analysis Method for fractional barrier option,PDE *J.Phys, : Conf. Ser. 2199*, pp.1-4.

[7] Faiz M.K Amjad A., Abdulah, Kamal S., Aziz K. and Ibrahi, M. (2022), Analytical Approximation of Brusselator Model via LADM, *Mathematical problems in Engineering*, volume 2022, pp. 1-14.

[8] Harish B. and Abhinandan C. (2019), Comparative analysis of numerical methods for the multidimensional Brusselator System,*Open Journal of Mathematical Sciences*, 3(2019), pp. 262-272.

[9] Hetmaniok E., Slota D., Trawinski T. and Witula R. (2014), Usage of the homotopy analysis method for solving the nonlinear and linear integral equations of the second kind,*Numer Algol* ,67 (1), pp.163 – 185.

[10] Higazy M., Sudhanshu A. and Taher A.N (2020), Sawi Decomposition method for Volterra Integral Equation with Application,*Hindawi Journal of Mathemtaics*, volume (2020), pp. 1-13.

[11] Higazy, M and Sudhanshu A. (2021), Sawi transformation for system of ordinary differential equations with application,*Ain Shams Engineering Journal*,12(2021), pp. 3173-3182.

[12] Jafari H., Abdelouahab K. and Baleanu D.S (2014), Variational Iteration Method for a Fractional-order Brusselator system , *Abstract and Applied Analysis*, volume (2014), pp. 1-6.

[13] Khani F., Samadi F. and Hamedi-Nezhad S. (2009), New Exact Solutions of the Brusselator Reaction –Diffusion Model Using Exp-Function Method, *Mathematical Problems in Engineering*, Volume (2009), pp. 1-9.

[14] Korkut S.ö and Cicek Y (2021), Numerical Solution of the Brusselator model by time Splitting method, *Cumhuriyet Sci. J.*; 42(1), pp.75-87.

[15] Li-Guo Y. and Ju-Hong K (2017), Stability and A numerical solution of Fractional-order Brusselator Chemical Reaction System, *Journal of Fractional Calculus and Applications*,8(1), pp.38-47.

[16] Mamta K. and Samanyu K.(2023), An iterative approach using Sawi transformfor fractional telegraph equation in diversified dimensions, *Nonlinear Engineering*, 2023(12), pp. 1-18.

[17] Matzinger E. (2006), Asymptotic behaviour of solutions near a turning point: the example of the Brusselator equation, *journal of Differential equations*, 220(2006), pp. 478-510.

[18] Mittal R.C and Jiwari R. (2011), Numerical Solution of two- Dimensional reaction-diffusion Brusselator system,*Applied Mathematics and computation*, 217(2011), pp. 5404-5415.

[19] Mohmed S.M., Khaled A.G., Muteb R.A and Refah A.A (2016), Homotopy Analysis transform method for integro-differential equation,*Gen. Math. Notes* , 32(1), pp. 32-48

[20] Mohand M.A.M (2019), The new integral transform “Sawi Transform”, *Advances in Theoretical and Applied Mathematics*, 14(1), pp.81-87.

[21] Mohanad F.K and Ali H.A (2022), Mixed Homotopy Integral Transform Method for Solving Integro-Differential Equation,*AL-Nahrain Journal of Science*, 25(1), pp.35-40.

[22] Patil D.D (2021a), Sawi transform and convolution theorem for initial Boundary value problem (wave equation), *Journal of Research & Development* ,11(14), pp. 1-4.

[23] Patil D.D (2021b), Application of Sawi transform of error function for Evaluating improper integral, *Journal of Research & Development*, 11(20), pp.1-6.

[24] Mohand M.A.M (2019), The New Integral Transform ”Sawi transform” *Advances in Theoretical and Applied Mathematics*,14(1), pp.81-87.

[25] Rabie S., Kharrat B.N., Joujeh G. and Abd Alulkader J. (2023), A new approach for solving boundary and initial value Problems by coupling the he method and Sawi transform, *Phys , Astron Int. J*, 7(2), pp. 141-144.

[26] Saad A.M., Rostam K.S and Fadhil H.E (2010), Numerical solution of Brusselator Model by finite different method, *journal of Applied Sciences Research*,6(11), pp.1632-1646.

[27] Salem A (2017), Third-order Approximate Solution of Chemical Reaction-Diffusion Brusselator System Using optimal Homotopy Asymptotic Method, *Advances in Mathematical Physics*,volume (2017), pp.1-8.

[28] Sahoo M. and Chakraverty S.(2022), Sawi transform Based Homotopy perturbation method for solving Shallow Water Wave Equations in Fuzzy Environment, *Mathematics*, 10(2900), pp.1-24.

[29] Singh J., Rashidi M.M., Kumar D. and Swroop R. (2016), A fractional model of a dynamic Brusselator reaction-diffusion system arising in triple collision and enzymatic reactions, *Nonlinear Engineering*, 5(4), pp. 277-285.

[30] Sudhanshu A. Swarg D.S and Aakansha V.(2020), Application of Sawi Transformation for Solving Convolution Type Volterra Integro-Differential Equation of First Kind,*International Journal of Latest Technology in Engineering, Management & Applied Science (IJLTEMAS)*, IX(VIII), pp. 1-8.