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Congruence Labeling of Some Graphs

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Abstract

Distribution of integers as labels to the vertices and edges of a graph with certain constrain is stated as graph labeling. In this paper, congruence and prime congruence labeling of some graphs have been introduced. A graph is identified as congruence graph and prime congruence graph if it concedes congruence labeling and prime Congruence labeling respectively. The propound Congruence labeling is recognized in path graph, cycle graph, friendship graph and cycle graph with zigzag chords. Further, fan graph, star graph and gear graph are arrived as prime congruence graph.

Keywords: Prime labeling, Congruence labeling, Prime Congruence labeling

INTRODUCTION

Graph theory was introduced by the Swiss Mathematician Leonhard Euler in 1736[3] to represent the Koingsberg bridge problem as a graph and arrived that there is no solution to the problem. Graph labeling have been introduced by A. Rosa in 1966[8]. Every mathematical fact can be represented as Graphs and labeled in different ways. In the recent years, labeling technique was reckoned as the most significant branch of graph theory, which influenced several mathematicians and gained numerous labeling concepts. Impressed by the presumption of labeling, Roger Entringer introduced prime labeling and it was explored by Tout.A, Dabbouncy.A.N and Howalla.K [10]. Labeling marches in innumerable sectors involving X-ray, crystallography, coding theory, circuit design, communication networks, astronomy, radar and data base management [9]. In this paper, the definition of Congruence and Prime Congruence labeling were established, which induces the labeling of vertices and edges by congruence modulo of integers. It is proved that path graph, cycle graph, friendship graph and cycle graph with zigzag chord are congruence graph. Also fan graph, star graph and gear graph are verified as prime congruence graph. Throughout this paper, simple undirected graphs were considered.

Preliminaries

Definition 2.1: [5] A graph $G = (V, E)$ is observed as *Prime graph* provided that there exist a bijection $g: V \longrightarrow \{1, 2, |V|\}$ and for every edge $e = x_1x_2$, $gcd (g(x_1), g(x_2)) = 1$.

Definition 2.2: [12] A wheel graph is named as *Gear graph* when a vertex is introduced between every pair of adjacent vertices of outer cycle and G_r has $2r + 1$ vertices and $3r$ edges.

Definition 2.3: [4] All the vertices of a Path is connected to a new vertex to yield a *fan graph* f_r and it has $r + 1$ vertices and $2r - 1$ edges.

Definition 2.4: [11] A graph ensued by attaching r copies of C_r with a common vertex is referred as *friendship graph* F_r.

Definition 2.5: [4] A *Star graph* S_r is a tree with one internal vertex of degree r-1 and degree one for the remaining vertices.

Definition 2.6: [7] A *cycle with zigzag chord*, is obtained from the cycle

C_r: v_0, v_1, v_{r-1}, v_0 for all $r \geq 8$ by adding the chords $v_1v_{r-1}, v_{r-1}v_3, v_\alpha v_\beta$, where

(i) $\alpha = \frac{r-2}{2}$ and $\beta = \frac{r+2}{2}$ for $r \equiv 0\,(mod\ \ 4)$

(ii) $\alpha = \frac{r-3}{2}$ and $\beta = \frac{r+3}{2}$ for $r \equiv 1\,(mod\ \ 4)$

(iii) $\alpha = \frac{r+4}{2}$ and $\beta = \frac{r}{2}$ for $r \equiv 2\,(mod\ \ 4)$

(iv) $\alpha = \frac{r+5}{2}$ and $\beta = \frac{r-1}{2}$ for $r \equiv 0\,(mod\ \ 4)$

Main Results

Definition 3.1: A graph G = (V,E) with r vertices and s edges is stated as *congruence graph* if there exist a vertex labeling $f : V \longrightarrow \{1, 2, ...k\}$ for every $x_i \in V$ and induces the edge labeling $g : E \longrightarrow \{1, 2, ...k - 1\}$ provided $f(x_{i+1}) \equiv f(x_i)\ (mod\ g\,(y_i))$, for every $y_i = x_ix_{i+1}$ where $k = min\,\{2r, 2s\}$.

Definition 3.2: A Congruence graph G = (V, E) is flourished as *prime congruence graph* if it satisfies $gcd\,\{f(x_i), f(x_{i+1})\} = 1$.

Theorem 3.3: Every path P_r is congruence graph, for $r \geq 2$.

Proof: Consider a path P_r with r vertices and $(r - 1)$ edges.

Let x_1, x_2, xn and y_1, y_2, yn_{-1} be the vertices and edges respectively.

$$\text{Let } k = min\,\{2r, (2r-2)\}$$
$$= 2r - 2$$

Define,

$$g\,(y_i) = i + 1$$

$$f(x_r) = k$$

$$f(x_{r-I}) = \begin{cases} |f(x_{n-i+1}) - y_{n-i}|, \; if\ i\ is\ odd \\ |f(x_{n-i+1}) + y_{n-i}|, \; if\ i\ is\ even \end{cases}$$

each $y \in E$ satisfies the relation $f(x_{i+1}) \equiv f(x_i)\ (mod\ g(y_i))$

Thus, the path P_r is congruence graph.

Example 3.4: Consider a path graph P_8 with 8 vertices and 7 edges,

Here, $k = 14$

The congruence labeling of P_8 is given in figure 1.

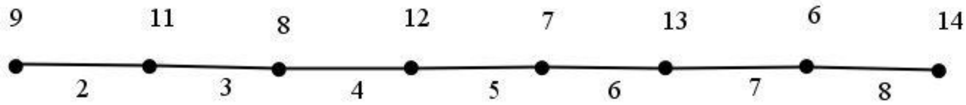

Figure: 1 - P_8 congruence graph

Theorem 3.5: The Cycle graph C_r is congruence graph, for $r \geq 3$.

Proof: C_r has r vertices and r edges

$$k = min\{2r, 2r\}$$
$$= 2r$$

The vertices of C_r are represented as x_1, x_2, xr,

Consider two cases,

If r is even, then the vertex labeling is defined as given below

$$f(x_i) = \begin{cases} 1 & for\ i = 1 \\ 2r & for\ i = r \\ 2r - \frac{i}{2} & for\ i = 2,4, \ldots r-2 \\ \frac{i+1}{2} & for\ i = 3,5, \ldots r-1 \end{cases}$$

If r is odd then the vertex are labeled in the following way,

$$f(x_i) = \begin{cases} 1 & for\ i = 1 \\ 2r & for\ i = r \\ 2r - \frac{i}{2} & for\ i = 2,4, \ldots r-1 \\ \frac{i+1}{2} & for\ i = 3,5, \ldots r-2 \end{cases}$$

$$g(y_i) = \begin{cases} |f(x_i) - f(x_{i+1})|, & for\ i \leq (r-1) \\ |f(x_{i+1}) - f(x_1)|, & for\ i = r \end{cases}$$

Hence, Cycle graph C_r is congruence graph.

Example 3.6: A cycle graph C_9 with 9 vertices and 9 edges was examined.

Here, $k = 18$

The congruence labeling of C_9 is given in figure 2.

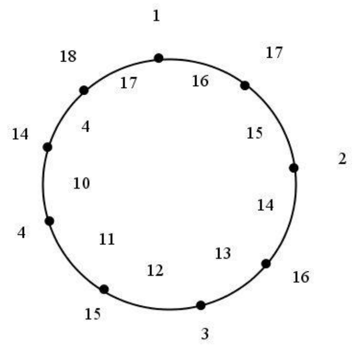

Figure 2: C_9 congruence graph

Theorem 3.7: The Friendship graph F_r is congruence graph.

Proof: Observe F_r with $(2r + 1)$ vertices and $3r$ edges

$$k = min\{2(2r+1), 2(3r)\}$$
$$= 4r + 2$$

x_1, x_2, x_{2n+1} are the vertices of F_r and x_1 is the vertex with degree $2r$

Define,

$$f(x_i) = \begin{cases} 1 & for\ i = 1 \\ i & for\ i = 2,4, \ldots 2r \\ k - i + 3 & for\ i = 3,5, \ldots 2r+1 \end{cases}$$

Then each edges are labeled with the difference of their adjacent vertex label

i.e, $g(y_i) = |f(x_{i+1}) - f(x_i)|$

Therefore, $f(x_{i+1}) \equiv f(x_i) \,(mod\ g(y_i))$

Example 3.8: A friendship graph F_4 with 9 vertices and 12 edges was verified

Here, $k = 18$

The congruence labeling of F_4 is displayed in figure 3.

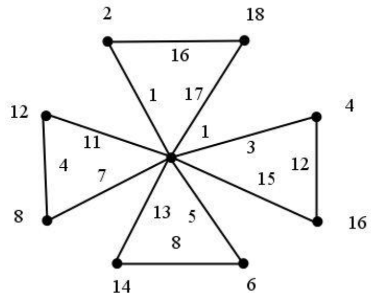

Figure 3 - F_4 congruence graph

Theorem 3.9: Every Cycle graph C_r with zigzag chords is congruence graph, for $r \geq 8$.

Proof: A Cycle graph with zigzag chords is represented as G.

To prove G is congruence, consider the four cases

- **(i)** $r \equiv 0 \,(mod\ 4)$
- **(ii)** $r \equiv 1 \,(mod\ 4)$
- **(iii)** $r \equiv 2 \,(mod\ 4)$
- **(iv)** $r \equiv 3 \,(mod\ 4)$

Case 1: For $r \equiv 0 \,(mod\ 4)$, $|V| = r$ and $|E| = (\frac{3r-2}{2})$.

Define,

$$f(x_0) = 1$$

$$f(x_{2i}) = i + 1, for\ 1 \leq i \leq \frac{r-4}{4}$$

$$f(x_{\frac{r+4i-4}{2}}) = i + 2, for\ 1 \leq i \leq \frac{r}{4}$$

$$f(x_{2i-1}) = \frac{3r + 2 - 2i}{2}, for\ 1 \leq i \leq \frac{r}{2}$$

Case 2: For $r \equiv 1 \,(mod\ 4)$, $|V| = r$ and $|E| = (\frac{3r-3}{2})$.

Define,

$$f(x_0) = 1$$

$$f(x_{2i}) = i + 1, for\ 1 \leq i \leq \frac{r-5}{4}$$

$$f(x_{\frac{r+4i-3}{2}}) = \frac{r + 4i - 1}{4}, for\ 1 \leq i \leq \frac{r-1}{4}$$

$$f(x_{2i-1}) = \frac{3r+1-2i}{2}, for\ 1 \le i \le \frac{r-1}{4}$$

$$f(x_{\frac{r+4i-1}{2}}) = \frac{5r+3-4i}{4}, for\ 1 \le i \le \frac{r-1}{4}$$

$$f(x_{\frac{r-1}{2}}) = \frac{3r+5}{4}$$

Case 3: For $r \equiv 2\,(mod\ 4)$, $|V|=r$ and $|E| = (\frac{3r-2}{2})$.

Let $r = 4p + 2\,(p \ge 2)$

If $p = 2$ then G has 14 edges.

$f(x_0) = 9$, $f(x_1) = 1$, $f(x_2) = 4$, $f(x_3) = 3$, $f(x_4) = 5$, $f(x_5) = 12$, $f(x_6) = 2$, $f(x_7) = 14$, $f(x_8) = 10$ and $f(x_9) = 15$.

For $p \ge 3$, label the vertices in the following way,

$$f(x_{r-1}) = 1$$

$$f(x_{2i+2}) = i+1, for\ 1 \le i \le \frac{r-6}{4}$$

$$f\left(x_{\frac{r-2i+6}{2}}\right) = \frac{r+4i-6}{4}, for\ 1 \le i \le 2$$

$$f(x_{\frac{r+4i+6}{2}}) = \frac{r+4i+10}{4}, for\ 1 \le i \le \frac{r-10}{4}$$

$$f(x_{r-2i-1}) = \frac{r+4+2i}{2}, for\ 1 \le i \le \frac{r-10}{4}$$

$$f(x_{\frac{r+6}{2}}) = \frac{3r+2}{4}$$

$$f(x_{2i-1}) = \frac{3r+2-2i}{2}, for\ 1 \le i \le \frac{r-2}{4}$$

$$f(x_{\frac{r}{2}}) = \frac{5r+10}{4}$$

$$f(x_{4-2i}) = r+5 = 3i\ for\ 1 \le i \le 2$$

Case 4: For $r \equiv 3\,(mod\ 4)$, $|V|=r$ and $|E| = (\frac{3r-3}{2})$.

Let $r = 4p + 3\,(p \ge 2)$ and the vertices are labeled as defined below,

$f(x_1) = 1$

$$f(x_{2i+1}) = i+1, for\ 1 \le i \le \frac{r+1}{4}$$

$$f\left(x_{\frac{r-4i+5}{2}}\right) = \frac{r+4i+9}{4}, for\ 1 \le i \le \frac{r-3}{4}$$

$$f(x_{r-2i}) = \frac{r+2i+3}{2}, for\ 1 \le i \le \frac{r-7}{4}$$

$$f(x_2) = r-1$$

$$f(x_0) = \frac{3r-1}{2}$$

$$f(x_{r-2i+1}) = \frac{3r-1-2i}{2}, for\ 1 \le i \le \frac{r-3}{4}$$

For all cases, edges are labeled as in theorem 3.3

Hence, every Cycle graph C_r with zigzag chords is congruence graph.

Example 3.10: Consider the cycle graph with zigzag chord for $16 \equiv 0\,(mod\ 4)$

Here, $k = 32$

The congruence labeling of C_{16} with zigzag chords is shown in figure - 4.

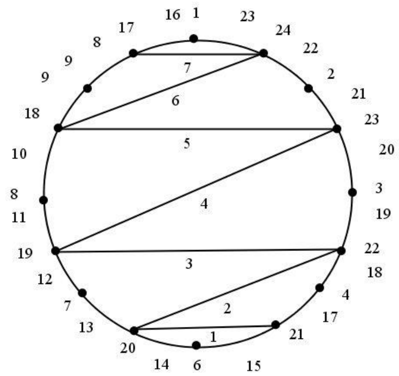

Figure 4: C_{16} with zigzag chords congruence graph

Theorem 3.11: The Star graph S_r is prime congruence graph, for $r \ge 2$.

Proof: Observe a Star graph S_r with $|V|=r$ and $|E| = (r-1)$

Let x_r be the center vertex of S_r and x_1, x_2, xr_{-1} be the pendent vertices of S_r

$$f(x_i) = \begin{cases} i+1, for\ i = 2,3,\dots r-1 \\ 1 \qquad for\ \ i = r \end{cases}$$

$$g(y_i) = i$$

x_r is the only vertex connected to pendent vertex, which is labeled as 1.

Therefore the $gcd\,\{f\,(x_r), f\,(x_i)\} = 1$, for $i = 2, 3, r-1$

Thus, the Star graph S_r is prime congruence graph.

Example 3.12: A prime congruence star graph S_{10} with 10 vertices and 11 edges is given infigure - 5

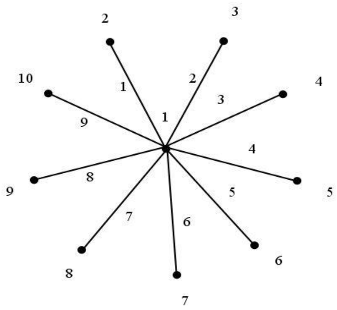

Figure 5: S_{10} prime congruence graph

Theorem 3.13: The Gear graph G_r is prime congruence graph if $r = 6$.

Proof: Consider G_r with $|V| = (2r + 1)$ and $|E| = 3r$

$$k = min\{2(2r + 1), 6r\}$$
$$= 4r + 2$$

Define,

$$f(x_i) = 1 \text{ if } i = 1$$

$$f(x_{2i}) = \begin{cases} 2i & if\ 1 \le i \le r-2 \\ 2i+2 & if\ i = 2r+2 \\ 2i+4 & if\ i = r \end{cases}$$

$$f(x_{2i-1}) = \begin{cases} k-2i+1 & if\ 2 \le i \le r-2 \\ k-2i-1 & if\ i = r-1 \\ k-2i+3 & if\ i = r \end{cases}$$

$$f(2r+1) = k-1$$

The vertices with odd degree are labeled with odd number and verticeswith even degree is labeled with even number,

Then $gcd\{f(x_{if}), f(x_j)\} = 1$

Thus, the Gear graph G_r for $r = 6$ is prime congruence graph.

Example 3.14: The prime congruence gear graph G_6 is verified in figure 6

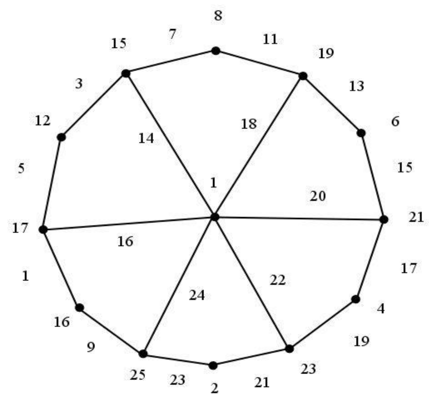

Figure 6: G_6 prime congruence graph

Theorem 3.15: The Fan graph f_r is prime congruence graph.

Proof: Suppose f_r is a fan graph with $|V| = (r+1)$ and $|E| = (2r-1)$ edges

Let u be the center vertex and x_1, x_2, xr are the leaf vertices of f_r

$$f_r = P_r + K_1$$

Here, $k = 2n + 2$

Label u as 1 and the rest of the vertices with 2, 3, $k-1$.

It satisfies $f(u) \equiv f(x_i)$ $(mod\ y)$ and $gcd\ \{f(u), f(x_i)\} = 1$ with distinct labels. Thus, the Fan graph f_n admits prime congruence labeling.

Example 3.16: A fan graph f_5 was established as prime congruence graph in the figure 7

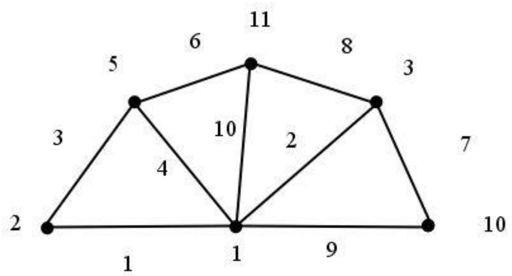

Figure 7: f_n congruence graph

CONCLUSION

A new kind of labeling such as Congruence labeling and Prime Congruence labeling have been introduced and examined the existence of Congruence labeling for path graph, cycle graph, friendship graph, cycle graph with zigzag chord. Also determined that fan graph, star graph and gear graph are Prime Congruence graph. In future, these concepts will be applied in communication network to enhance security network.

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