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Coefficient estimate of Bi-Univalent Function with respect to symmetric points Associated with Balancing Polynomials

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Abstract

In this paper, we introduce and investigate new subclasses of bi-univalent functions with respect to the symmetric points in $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ defined by Balancing polynomials. We obtain upper bounds for Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$ and Fekete-Szeg$\ddot{o}$ inequalities $|a_3 - \mu a_2^2|$ for these new subclasses
Keywords: Balancing polynomial;bi-univalent;subordination; function;analytic function;Taylor-Maclaurin coefficient,Fekete-Szeg$\ddot{o}$ functional.
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1. INTRODUCTION

Let the class of Analytic function in $\mathcal{U} = \{z \in \mathbb{C} : |Z| < 1\}$, denoted by $\mathcal{A}$, contain all the functions of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (z \in \mathcal{U}) \tag{1.1}$$

which satisfy the usual normalization condition $f(0) = f^{(}0) - 1 = 0$.
Let $\mathcal{S}$ be the class of $\mathcal{A}$ consist of all functions $f \in \mathcal{A}$, which are also univalent in $\mathcal{U}$. The Koebe one quarter theorem [1] ensure that the image of $\mathcal{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk radius of $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, (z \in \mathcal{U}) \text{ and } f(f^{-1}(w)) = w, \left(|w| < r_0(f), r_0 \geq \tfrac{1}{4}\right)$$

If f and f^{-1} are univalent in $\mathcal{U}$, the $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$, and the class of bi-univalent functions defined in the unit disk denoted by $\sum$. Sine $f \in \sum$ has the maclaurin series given by (1.1), a computations show that $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots \tag{1.2}$$

The expression $\sum$ is a non-empty class functions, as it contains at least the functions

$$f_1(z) = \frac{z}{1-z}, f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z}$$

with their corresponding inverse

$$f_1^{-1}(w) = \frac{w}{1-w}, f_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1}$$

In addition the Koebe function $f(z) = \frac{z}{(1-z)^2} \notin \sum$

The study of analytical and bi-univalent functions is reintroduced in the publication of [2] and is followed by work such as [3, 4, 5, 6, 7, 8]. The initial coefficients constraints have been determined by several authors who have also presented new sub class of bi-univalent functions [2, 3, 4, 6, 7, 9, 10, 11]

Consider α and β to be analytic function $\mathcal{U}$. We say that α subordinate β, if a Schwartz function w exist that is analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1, (z \in \mathcal{U})$

$$\alpha(z) = \beta(w(z)), (z \in \mathcal{U}).$$

This subordination is denoted by $\alpha \prec \beta$, or $(z \in \mathcal{U})$. Given that β is univalent in $\mathcal{U}$, then

$$\alpha(z) \prec \beta(z) \Leftrightarrow \alpha(0) = \beta(0) \text{ and } \alpha(\mathcal{U}) \subset \beta(\mathcal{U})$$

Using Loewner's technique, the Fekete-Szeg $\ddot{o}$ problem for the coefficients of $f \in \mathcal{S}$ in [6] is

$$|a_3 - \mu a_2^2| \le 1 + 2\exp\left(\tfrac{-2\mu}{1-\mu}\right) \text{ for } 0 \le \mu < 1$$

The elementary inequality $|a_3 - a_2^2| \le 1$ is obtained as $\mu \to 1$. The coefficient functional

$$F_\mu(f) = a_3 - \mu a_2^2$$

on the normalized analytic function f in the open unit disk $\mathcal{U}$ also has a significant impact on geometric function theory. The Fekete-Szeg$\ddot{o}$ problem is known as the maximization problem for the functional $|F_\mu(f)|$

Researchers were concerned about several classes of univalent functions [12, 13, 14, 15] due to the Fekete-Szeg$\ddot{o}$ problem, proposed in 1933 [16] ; therefore, it stands to reason that similar inequalities were also discovered for bi-univalent functions, and fairly recent publications can be cited to back up the claim that the subject still yields intriguing findings [17, 18, 19]
Because of their importance in probability theory, mathematical statistics, mathematical physics, and engineering, orthogonal polynomials have been the subject of substantial research in recent years from a variety of angles. The classical orthogonal polynomials are the orthogonal polynomials that are most commonly used in applications (Hermite polynomials, Laguerre polynomials, Jacobi polynomials, and Bernoulli). We point out [17, 18, 20, 21, 22, 23, 24] as more recent examples of the relationship between geometric function theory and classical orthogonal polynomial.
In the literature, there are many integer number sequences such as Fibonacci, Lucas, Pell and so on. Recently, Behera and Panda [25] introduced a new integer sequence named Balancing numbers. In the last quarter century, on some properties of this new number sequence have been intensively studied and its some generalizations were defined. Interested readers can find comprehensive information regarding Balancing numbers in [26, 27, 28, 29, 30, 31, 32, 33] and references therein. A natural generalization the Balancing numbers is the Balancing polynomials and its definition and some interesting properties can be found in [34]

The Balancing polynomials $B_n(x)$ is given by [35] using the generating function

$$B(x,z)=\sum_{n=0}^{\infty}B_n(x)z^n=\frac{z}{1-6xz+z^2} \tag{1.3}$$

The balancing polynomial are easily computed by the recurrence relation

$$B_n(x)=6xB_{n-1}(x)-B_{n-2}(x), \tag{1.4}$$

where

$$\begin{aligned} B_0(x)&=0\\ B_1(x)&=1\\ B_2(x)&=6x\\ B_3(x)&=36x^2-1 \end{aligned} \tag{1.5}$$

Sakaguchi [36] introduced the class S_s^* s of functions starlike with respect to symmetric points, which consists of functions $f \in S$ satisfying the condition

$$Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0(z \in \mathcal{U})$$

In addition, Wang et al.[37] introduced the class C_s of functions convex with respect to symmetric points, which consists of functions $f \in S$ satisfying the condition

$$Re\left\{\frac{[zf'(z)]'}{[f(z)-f(-z)]'}\right\} > 0(z \in \mathcal{U})$$

In this paper, we consider two subclasses of $\sum$ the class $S_S^{\sum}(x)$ of functions bi-starlike with respect to the symmetric points and the relative class $C_S^{\sum}(x)$ of functions bi-convex with respect to the symmetric points associated with Balancing polynomials. The definitions are as follows:

Definition 1.1. $f \in S_s^{\sum}(x)$, if the next subordination hold:

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec G(x,z) \tag{1.6}$$

and

$$\frac{2wg'(w)}{g(w)-g(-w)} \prec G(x,z) \tag{1.7}$$

where $z, w \in \mathbb{U}, B(x,z)$ is given by in (1.3) and $g = f^{-1}$ is given by (1.2)

Definition 1.2. $f \in C_s^{\sum}(x)$, if the next subordination hold:

$$\frac{2[zf'(z)]'}{f(z)-f(-z)} \prec G(x,z) \tag{1.8}$$

and

$$\frac{2[wg'(w)]'}{[g(w)-g(-w)]'} \prec G(x,z) \tag{1.9}$$

where $z, w \in \mathbb{U}, B(x,z)$ is given by in (1.3) and $g = f^{-1}$ is given by (1.2)

Lemma 1.3. *[38] Suppose that* $c(z) = \sum_{n=1}^{\infty} c_n z^n, |c(z)| < 1, z \in \mathcal{U}$ *is an analytic function in* $\mathcal{U}$*. Then,*

$$|c_1| \leq 1, |c_n| \leq 1 - |c_1|^2, n = 2, 3 \cdots$$

2. COEFFICIENT ESTIMATE FOR THE CLASS $S_s^{\sum}(x)$

We obtained upper bounds of $|a_2|$ and $|a_3|$ for the function belonging to the class $S_s^{\sum}(x)$.

Theorem 2.1. *If $f \in S_s^{\sum}(x)$ then*

$$|a_2| \leq \frac{|B_2(x)|\sqrt{|B_2(x)|}}{\sqrt{|2B_2^2(x) - 4B_3(x)|}} \tag{2.1}$$

and

$$|a_3| \leq \frac{|B_2(x)|}{2} + \frac{|(B_2(x))^2|}{4} \tag{2.2}$$

Proof. Let $f \in S_s^{\sum}(x)$ and $g = f^{-1}$ from definition (1.1) and (1.2), we have

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec G(x, \Phi(z)) \tag{2.3}$$

and

$$\frac{2wg'(w)}{g(w) - g(-w)} \prec G(x, \Psi(w)) \tag{2.4}$$

where Φ and Ψ are analytic in $\mathcal{U}$ is given by

$$\Phi(z) = \Phi_1 z + \Phi_2 z^2 + \cdots \tag{2.5}$$

and

$$\Psi(z) = \Psi_1 w + \Psi_2 w^2 + \cdots \tag{2.6}$$

and $\Phi(0) = \Psi(0) = 0$, and $|\Phi(z)| < 1, |\Psi(w)| < 1, z, w \in \mathcal{U}$
As a result of Lemma (1.3)

$$|\Phi_i| \leq 1 \;\; and \;\; |\Psi_i| \leq 1, i \in \mathbb{N} \tag{2.7}$$

If we replace (2.5) and (2.6) in (2.3) and (2.4), respectively, we obtain

$$\frac{2zf'(z)}{f(z) - f(-z)} = B_0(x) + B_1(x)\Phi(z) + B_2(x)\Phi^2(z) + \cdots \tag{2.8}$$

and

$$\frac{2wg'(w)}{g(w) - g(-w)} = B_0(x) + B_1(x)\Psi(z) + B_2(x)\Psi^2(z) + \cdots \tag{2.9}$$

In view of (1.1) and (1.2)from (2.8) and (2.9)

$$1 + 2a_2 z + 2a_3 z^2 + \cdots = B_1(x) + [B_2(x)\Phi_1]z + [B_2(x)\Phi_2 + B_3(x)\Phi_1^2]z^2 + \cdots \tag{2.10}$$

and

$$1-2a_2w+(4a_2^2-2a_3)w^2+\cdots = B_1(x)+[B_2(x)\Psi_1]w+[B_2(x)\Psi_2+B_3(x)\Psi_1^2]w^2+\cdots \tag{2.11}$$

which yields the following relations:

$$2a_2 = B_2(x)\Phi_1 \tag{2.12}$$

$$-2a_2 = B_2(x)\Psi_1 \tag{2.13}$$

$$2a_3 = B_2(x)\Phi_2 + B_3(x)\Phi_1^2 \tag{2.14}$$

$$4a_2^2 - 2a_3 = B_2(x)\Psi_2 + B_3(x)\Psi_1^2 \tag{2.15}$$

from (2.12) and (2.13) it follows that

$$\Phi_1 = -\Psi_1 \tag{2.16}$$

and

$$8a_2^2 = (\Phi_1^2 + \Psi_1^2)(B_2(x))^2$$

$$a_2^2 = \frac{(\Phi_1^2 + \Psi_1^2)(B_2(x))^2}{8} \tag{2.17}$$

Adding (2.14) and (2.15) using 2.17 we obtain

$$a_2^2 = \frac{(B_2(x))^3(\Phi_2 + \Psi_2)}{4[(B_2(x))^2 - 2B_3(x)]} \tag{2.18}$$

Using relation (1.5)from (2.7) for Φ_2 and Ψ_2, we get (2.1). Using 2.16 and 2.17, by subtracting 2.14 from 2.15 we get

$$\begin{aligned} a_3 &= \frac{B_2(x)(\Phi_2 - \Psi_2) - B_3(x)(\Phi_1^2 - \Psi_1^2)}{4} + a_2^2 \\ &= \frac{B_2(x)(\Phi_2 - \Psi_2) - B_3(x)(\Phi_1^2 - \Psi_1^2)}{4} + \frac{(\Phi_1^2 + \Psi_1^2)(B_2(x))^2}{8} \end{aligned} \tag{2.19}$$

again applying (2.16) and using 1.4 for the coefficients $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ we deduce (2.2) □

3. THE FEKETE-SZEG$\ddot{o}$ PROBLEM FOR THE FUNCTION CLASS $S_s^{\sum}$

We obtain the Fekete-Szeg$\ddot{o}$ inequality for the class $S_s^{\sum}$ due to the result of [19]

Theorem 3.1. *If f given by (1.1) is in the class $f \in S_s^{\sum}$ where $\mu \in \mathbb{R}$, then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2(x)}{2}, & \text{if } |\chi(\mu)| \leq \frac{1}{4} \\ 2B_2(x)|\chi(\mu)|, & \text{if } |\chi(\mu)| \geq \frac{1}{4} \end{cases}$$

where

$$\chi(\mu) = \frac{(1-\mu)(B_2(x))^2}{4[(B_2(x))^2 - 2B_3(x)]}$$

Proof. if $f \in S_s^{\sum}$ is given by (1.1) from (2.18) and (2.19) we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{B_2(x)(\Phi_2 - \Psi_2)}{4} + (1-\mu)a_2^2 \\ &= \frac{B_2(x)(\Phi_2 - \Psi_2)}{4} + \frac{(1-\mu)(B_2(x))^3(\Phi_2 + \Psi_2)}{4[(B_2(x))^2 - 2B_3(x)]} \\ &= B_2(x)\left[\frac{\Phi_2}{4} - \frac{\Psi_2}{4} + \frac{(1-\mu)(B_2(x))^2\Phi_2}{4[(B_2(x))^2 - 2B_3(x)]} + \frac{(1-\mu)(B_2(x))^2\Psi_2}{4[(B_2(x))^2 - 2B_3(x)]}\right] \\ &= B_2(x)\left[\left(\chi(\mu) + \frac{1}{4}\right)\Phi_2 + \left(\chi(\mu) - \frac{1}{4}\right)\Psi_2\right] \end{aligned}$$

where

$$\chi(\mu) = \frac{(1-\mu)(B_2(x))^2}{4[(B_2(x))^2 - 2B_3(x)]}$$

Now using (1.5)

$$a_3 - \mu a_2^2 = 6x\left[\left(\chi(\mu) + \frac{1}{4}\right)\Phi_2 + \left(\chi(\mu) - \frac{1}{4}\right)\Psi_2\right]$$

where

$$\chi(\mu) = \frac{(1-\mu)9x^2}{2(1-18x^2)}$$

Therefore, given (1.5)and (2.7), we conclude that necessary inequality holds. □

4. COEFFICIENT ESTIMATES FOR THE CLASS $C_s^{\sum}(x)$

We will obtain upper bound of $|a_2|$ and $|a_3|$ for the function class belonging to a class $C_s^{\sum}(x)$

Theorem 4.1. *$If f \in C_s^{\sum}(x)$, then*

$$|a_2| \leq \frac{|B_2(x)|\sqrt{|B_2(x)|}}{\sqrt{|6(B_2)^2(x) - 16B_3(x)|}} \tag{4.1}$$

and

$$|a_3| \leq \frac{|B_2(x)|}{6} + \frac{|(B_2(x))^2|}{16} \tag{4.2}$$

Proof. let $f \in C_s^{\sum}(x)(x)$ and $g = f^{-1}$. from (1.8) and (1.9), we get

$$\frac{2[zf'(z)]'}{f(z)-f(-z)} \prec G(x,\Phi(z)) \tag{4.3}$$

and

$$\frac{2[wg'(w)]'}{[g(w)-g(-w)]'} \prec G(x,\Psi(z)) \tag{4.4}$$

where Φ and Ψ are analytic function $\mathcal{U}$ given by

$$\Phi(z) = \Phi_1 z + \Phi_2 z^2 + \cdots \tag{4.5}$$

and

$$\Psi(z) = \Psi_1 w + \Psi_2 w^2 + \cdots \tag{4.6}$$

and $\Phi(0) = \Psi(0) = 0$, and $|\Phi(z)| < 1, |\Psi(w)| < 1, z, w \in \mathcal{U}$
As a result of Lemma (1.3)

$$|\Phi_i| \le 1 \ \ and \ \ |\Psi_i| \le 1, i \in \mathbb{N} \tag{4.7}$$

If we replace (4.5)and (4.6), in (4.3)and (4.4), respectively we obtain

$$\frac{2[zf'(z)]'}{f(z)-f(-z)} = B_0(x) + B_1(x)\Phi(z) + B_2(x)\Phi^2(z) + \cdots \tag{4.8}$$

and

$$\frac{2[wg'(w)]'}{[g(w)-g(-w)]'} = B_0(x) + B_1(x)\Psi(z) + B_2(x)\Psi^2(z) + \cdots \tag{4.9}$$

In view of (1.1) and (1.2) from (4.8) and (4.9) we obtain

$$1 + 4a_2 z + 6a_3 z^2 + \cdots = B_1(x) + [B_2(x)\Phi_1]z + [B_2(x)\Phi_2 + B_3(x)\Phi_1^2]z^2 + \cdots$$

and

$$1-4a_2w+(12a_2^2-6a_3)w^2+\cdots = B_1(x)+[B_2(x)\Psi_1]w+[B_2(x)\Psi_2+B_3(x)\Psi_1^2]w^2+\cdots$$

which yields the following relations:

$$4a_2 = B_2(x)\Phi_1 \tag{4.10}$$

$$-4a_2 = B_2(x)\Psi_1 \tag{4.11}$$

$$6a_3 = B_2(x)\Phi_2 + B_3(x)\Phi_1^2 \tag{4.12}$$

$$12a_2^2 - 6a_3 = B_2(x)\Psi_2 + B_3(x)\Psi_1^2 \tag{4.13}$$

From (4.10) and (4.10) it follows that

$$\Phi_1 = -\Psi_1 \tag{4.14}$$

and

$$\begin{aligned} 32a_2^2 &= [(B_1)(x)]^2(\Phi_1^2 + \Psi_1^2) \\ a_2^2 &= \frac{[(B_1)(x)]^2(\Phi_1^2 + \Psi_1^2)}{32} \end{aligned} \tag{4.15}$$

Adding (4.12) and (4.13) using (4.15), we obtain

$$\frac{B_2^3(x)(\Phi_1 + \Psi_1)}{4(3[B_2(x)]^2 - 8B_3(x))} \tag{4.16}$$

Using relation (1.5), (4.7) for Ψ_2,Ψ_2 we get (4.1). Using (4.14) and (4.15), by subtracting (4.13) from relation (4.12), we get

$$\begin{aligned} a_3 &= \frac{B_2(x)(\Phi_2 - \Psi_2) + B_3(x)(\Phi_1^2 - \Psi_1^2)}{12} + a_2^2 \\ &= \frac{B_2(x)(\Phi_2 - \Psi_2) + B_3(x)(\Phi_1^2 - \Psi_1^2)}{12} + \frac{[(B_1)(x)]^2(\Phi_1^2 + \Psi_1^2)}{32} \end{aligned} \tag{4.17}$$

Once again applying (4.14) and using (1.5) for coefficients Φ_1,Φ_2,Ψ_2,Ψ_2 we deduce (4.2). □

5. THE FEKETE-SZEG$\ddot{o}$ PROBLEM FOR THE FUNCTION CLASS $C_s^{\sum}$(X)

We obtain the Fekete-Szeg$\ddot{o}$ inequality for the class $C_s^{\sum}$ due to the result of [19]

Theorem 5.1. *If f given by (1.1) is in the class $f \in S_s^{\sum}$ where $\mu \in \mathbb{R}$, then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2(x)}{6}, & \text{if } |\chi(\mu)| \leq \frac{1}{12} \\ 2B_2(x)|\chi(\mu)|, & \text{if } |\chi(\mu)| \geq \frac{1}{12} \end{cases}$$

where

$$\chi(\mu) = \frac{(1-\mu)(B_2(x))^2}{4[3(B_2(x))^2 - 8B_3(x)]}$$

Proof. if $f \in S_s^{\sum}$ is given by (1.1) from (2.18) and (2.19) we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{B_2(x)(\Phi_2 - \Psi_2)}{12} + (1-\mu)a_2^2 \\ &= \frac{B_2(x)(\Phi_2 - \Psi_2)}{12} + \frac{(1-\mu)(B_2(x))^3(\Phi_2 + \Psi_2)}{4[3(B_2(x))^2 - 8B_3(x)]} \\ &= B_2(x)\left[\frac{\Phi_2}{12} - \frac{\Psi_2}{12} + \frac{(1-\mu)(B_2(x))^2\Phi_2}{4[3(B_2(x))^2 - 8B_3(x)]} + \frac{(1-\mu)(B_2(x))^2\Psi_2}{4[3(B_2(x))^2 - 8B_3(x)]}\right] \\ &= B_2(x)\left[\left(\chi(\mu) + \frac{1}{12}\right)\Phi_2 + \left(\chi(\mu) - \frac{1}{12}\right)\Psi_2\right] \end{aligned}$$

where

$$\chi(\mu) = \frac{(1-\mu)(B_2(x))^2}{4[3(B_2(x))^2 - 8B_3(x)]}$$

Now using (1.5)

$$a_3 - \mu a_2^2 = 6x\left[\left(\chi(\mu) + \frac{1}{12}\right)\Phi_2 + \left(\chi(\mu) - \frac{1}{12}\right)\Psi_2\right]$$

where

$$\chi(\mu) = \frac{(1-\mu)9x^2}{4(2-45x^2)}$$

Therefore, given (1.5)and (4.7), we conclude that necessary inequality holds. □

6. CONCLUSION

We introduce and investigate new subclasses of bi-univalent functions in $\mathcal{U}$ associated with Balancing polynomials and satisfying subordination conditions. Moreover, we obtain upper bounds for the initial Taylor-Maclaurin coeffcients $|a_2|$, $|a_3|$ and Fekete-Szeg$\ddot{o}$ problem $|a_3 - \mu a_2^2|$ for functions in these subclasses.

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