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Fixed Point Theorems for $\beta-\psi-\varphi-$Contractive Type Mappings in non- Newtonian Metric Spaces

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Abstract

In this paper we introduced the concept of $\beta-\psi-\varphi-$contractive type mappings in the setting of non-Newtonian metric space. We also provide some example to illustrate our result.

INTRODUCTION

Fixed point theory is one of the most dynamic research topic in non linear analysis and many fruitful results have come into the literature in the last few decades. The most remarkable result was given by **Banach [1]** in 1922 as Banach contraction principle. Later on, many generalization of Banach Contraction Principle came into existence in the literature [2-4]. **Samet *et al.* [5]** introduced the notion of α-ψ-contractive mappings and proved the related fixed point theorems.

The study of non-Newtonian calculi has been started in 1972 by **Grossman and Katz [6]**. These provide an alternative to the classical calculus and they include the geometric, anageometric and bigeometric calculi, etc. In 2002, **Cakmac and Basar [7]**, introduced the concept of non-Newtonian metric space. Recently, **Binbasıoglu *et. al.* [8]** discussed some topological properties of the non-Newtonian metric space and also introduced the contraction principle in non-Newtonian metric space.

In this paper, we generalize the concept of α-ψ mappings as $\beta-\psi-\varphi-$contractive mappings in the setting of non- Newtonian metric spaces.

PRELIMINARIES

Now, we define the non-Newtonian real field and we give the relevant properties due to **Cakmak and Basar [7]**.

A *generator* is defined as an injective map $\alpha: \mathbb{R} \to \mathbb{R}$ where the range is a subset of $\mathbb{R}$.

The necessary and sufficient condition that each generator generates one arithmetic is that each arithmetic is generated by one generator.

If $\alpha = I, I(x) = x \; \forall \; x \in \mathbb{R}$, then α generates the classical arithmetic. If $\alpha = exp, exp(x) = e^x \forall x \in \mathbb{R}$, then α generates geometrical arithmetic.

The set $\mathbb{R}(N)$ is defined as

$$\mathbb{R}(N) \coloneqq \{\alpha(x) : x \in \mathbb{R}\},$$

and $\mathbb{R}(N)$ is said to be the set of non-Newtonian real numbers.

All concepts of α-arithmetic have similar properties in classical arithmetic. α-zero, α-one and other α-integers are formed as

$$\dots, \alpha(-1), \alpha(0), \alpha(1), \dots.$$

Let α be any generator with range A. Then, the operations α-addition, α -subtraction, α -multiplication, α -division and α -order are defined in the following way for $x, y \in \mathbb{R}$, respectively:

α-addition $\quad x \dot{+} y = \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\}$,

α–subtraction $\quad x \dot{-} y = \alpha\{\alpha^{-1}(x) - \alpha^{-1}(y)\}$,

α –multiplication $\quad x \dot{\times} y = \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(y)\}$,

α –division $\quad x \dot{/} y = \alpha\{\alpha^{-1}(x) \div \alpha^{-1}(y)\}$,

α–order $\quad x \dot{<} y \Leftrightarrow \alpha(x) < \alpha(y)$.

For $x \in A \subset \mathbb{R}(N)$, a number α -square is described by $x \dot{\times} x$ and denoted by x^{2_N}. The symbol $\sqrt{x}^N$ denotes

$$t = \alpha\left\{\sqrt{\alpha^{-1}(x)}\right\}$$

which is the unique α nonnegative number whose α -square is equal to x and which means $t^{2_N} = x$, for each α nonnegative number t. Throughout this paper, x^{p_N} denotes the pth non-Newtonian exponent. Thus we have

$$x^{p_N} = x^{(p-1)_N} \dot{\times} x = \alpha\{[\alpha^{-1}(x)]^p\},$$

We denote by $|x|_N$ the α -absolute value of a number $x \in A \subset \mathbb{R}(N)$ defined as $\alpha(|\alpha^{-1}(x)|)$ and also

$$\sqrt{x^{2_N}}^N = |x|_N = \alpha\{|\alpha^{-1}(x)|\}$$

Thus,

$$|x|_N = \begin{cases} x, & x \dot{>} \alpha(0), \\ \alpha(0), & x = \alpha(0), \\ \alpha(0) \dot{-} x, & x \dot{<} \alpha(0). \end{cases}$$

For $x_1, x_2 \in A \subseteq \mathbb{R}(N)$, the non-Newtonian distance $|\cdot|_N$ is defined as

$$|x_1 \dot{-} x_2|_N = \alpha\{|\alpha^{-1}(x_1) - \alpha^{-1}(x_2)|\}.$$

This distance is commutative; i.e., $|x_1 \dot{-} x_2|_N = |x_2 \dot{-} x_1|_N$.

Take any $z \in \mathbb{R}(N)$, if $z \dot{>} \beta(0)$, then z is called a positive non-Newtonian real number; if $z \dot{<} \alpha(0)$, then z is called a non-Newtonian negative real number and if $z = \alpha(0)$, then z is called an unsigned non-Newtonian real number. Non-Newtonian positive real numbers are denoted by $\mathbb{R}^+(N)$ and non-Newtonian negative real numbers by $\mathbb{R}^-(N)$ **[6]**.

The fundamental properties that are provided in the classical calculus are also provided in non-Newtonian calculus, too.

Proposition 2.9 [7]. The triangle inequality with respect to non-Newtonian distance $|\cdot|_N$, for any $x, y \in \mathbb{R}(N)$ is given by $|x \dot{+} y|_N \dot{\leq} |x|_N \dot{+} |y|_N$.

Definition 2.10 [7]. Let $X \neq \emptyset$ be a set. If a function $d_N: X \times X \to \mathbb{R}^+(N)$ satisfies the following axioms for all $x, y, z \in X$:

(NM1) $d_N(x, y) = \alpha(0) = \dot{0}$ if and only if $x = y$,

(NM2) $d_N(x, y) = d_N(y, x)$,

(NM3) $d_N(x, y) \dot{\leq} d_N(x, z) \dot{+} d_N(z, y)$,

then it is called a non-Newtonian metric on X and the pair (X, d_N) is called a non-Newtonian metric space.

Definition 2.11 [8] Let (X, d_N) be a non-Newtonian metric space, $x \in X$ and $\varepsilon \dot{>} \dot{0}$, we now define a set $B_\varepsilon^N(x) = \{y \in X : d_N(x, y) \dot{<} \varepsilon\}$, which is called a non-Newtonian open ball of radius ε with center x. Similarly, one describes the non-Newtonian closed ball as $\bar{B}_\varepsilon^N(x) = \{y \in X : d_N(x, y) \dot{\leq} \varepsilon\}$.

Example 2.12 Consider the non-Newtonian metric space $(\mathbb{R}^+(N), d_N^*)$. From the definition of d_N^*, we can verify that the non-Newtonian open ball of radius $\varepsilon \dot{<} \dot{1}$ with center x_0 appears as $(x_0 \dot{-} \varepsilon, x_0 \dot{+} \varepsilon) \subset \mathbb{R}^+(N)$.

Definition 2.13 [8] Let (X, d_X^N) and (Y, d_Y^N) be two non-Newtonian metric spaces and let $f : X \to Y$ be a function. If f satisfies the requirement that, for every $\varepsilon \dot{>} \dot{0}$, there exists $\delta \dot{>} \dot{0}$ such that $f(B_\delta^N(x)) \subset B_\varepsilon^N(f(x))$, then f is said to be non-Newtonian continuous at $x \in X$.

Example 2.14 Given a non-Newtonian metric space (X, d_N), define a non-Newtonian metric on $X \times X$ by $p((x_1, x_2), (y_1, y_2)) = d_N(x_1, y_1) \dot{+} d_N(x_2, y_2)$. Then the non-Newtonian metric $d_N : X \times X \to (\mathbb{R}^+(N), |\cdot|_N)$ is non-Newtonian continuous on $X \times X$. To show this, let $(y_1, y_2), (x_1, x_2) \in X \times X$.

Since we have $|d_N(y_1, y_2) \dot{-} d_N(x_1, x_2)|_N \dot{\leq} d_N(x_1, y_2) \dot{+} d_N(x_2, y_2)$, it is clear that d_N is non-Newtonian continuous on $X \times X$. Now, we emphasize some properties of convergent sequences in a non Newtonian metric space.

Definition 2.15 [8] A sequence (x_n) in a metric space $X = (X, d_N)$ is said to be convergent if for every given $\varepsilon \dot{>} \dot{0}$ there exist an $n_0 = n_0(\varepsilon) \in Nand\ x \in X$ such that $d_N(x_n, x) \dot{<} \varepsilon$ for all $n > n_0$, and it is denoted by ${}^N\lim_{n\to\infty} x_n = x$ or $x_n \overset{N}{\to} x$, as $n \to \infty$.

Definition 2.16 [8] A sequence (x_n) in a non-Newtonian metric space $X = (X, d_N)$ is said to be non-Newtonian Cauchy if for every $\varepsilon \dot{>} \dot{0}$ there exists an $n_0 = n_0(\varepsilon) \in N$ such that $d_N(x_n, x_m) \dot{<} \varepsilon$ for all $m, n > n_0$. Similarly, if for every non-Newtonian open ball $B_\varepsilon^N(x)$, there exists a natural number n_0 such that $n > n_0$, $x_n \in B_\varepsilon^N(x)$, then the sequence (x_n) is said to be non-Newtonian convergent to x.

The space X is said to be non-Newtonian complete if every non-Newtonian Cauchy sequence in X converges **[7]**.

Proposition 2.17 [7] Let $X = (X, d_N)$ be a non-Newtonian metric space. Then

(i) a convergent sequence in X is bounded and its limit is unique,

(ii) a convergent sequence in X is a Cauchy sequence in X.

Lemma 2.18 [7] Let (X, d_N) be a non-Newtonian metric space, (x_n) a sequence in X and $x \in X$. Then $x_n \overset{N}{\rightarrow} x \ (n \rightarrow \infty)$ if and only if $d_N(x_n, x) \overset{N}{\rightarrow} \dot{0} \ (n \rightarrow \infty)$.

Lemma 2.19 [8] Let (X, d_N) be a non-Newtonian metric space and let (x_n) be a sequence in X. If the sequence (x_n) is non-Newtonian convergent, then the non-Newtonian limit point is unique.

Theorem 2.20 [8] Let (X, d_N^X) and (Y, d_N^Y) be two non-Newtonian metric spaces, $f : X \rightarrow Y$ a mapping and (x_n) any sequence in X. Then f is non-Newtonian continuous at the point $x \in X$ if and only if $f(x_n) \overset{N}{\rightarrow} f(x)$ for every sequence (x_n) with $x_n \overset{N}{\rightarrow} x \ (n \rightarrow \infty)$.

Definition 2.21 [8] Let X be a set and T a map from X to X. A fixed point of T is a point $x \in X$ such that $Tx = x$. In other words, a fixed point of T is a solution of the functional equation $Tx = x, x \in X$.

Definition 2.22 [8] Suppose that (X, d_N) is a non-Newtonian complete metric space and $T : X \rightarrow X$ is any mapping. The mapping T is said to satisfy a non-Newtonian Lipschitz condition with $k \in \mathbb{R}(N)$ if $d_N(T(x), T(y)) \dot{\leq} k \dot{\times} d_N(x, y)$ holds for all $x, y \in X$.

If $k \dot{<} \dot{1}$, then T is called a non-Newtonian contraction mapping.

Theorem 2.23 [8] Let T be a non-Newtonian contraction mapping on a non Newtonian complete metric space X. Then T has a unique fixed point.

MAIN RESULTS

Definition 3.1 Let (X, d_N) be a non-Newtonian metric space and $T: X \rightarrow X$ be a mapping. T is a non-Newtonian$-\beta - \psi - \varphi$ contractive type mapping if there exist three functions $\beta: X \times X \rightarrow \mathbb{R}^+(N)$ and $\psi, \varphi \in \Psi$ such that

$$\beta(x, y) \dot{\times} \psi\big(d(Tx, Ty)\big) \dot{\leq} \psi\big(d(x, y)\big) \dot{-} \varphi\big(d(x, y)\big) \qquad (1)$$

Theorem 3.2: Let (X, d_N) be a complete non-Newtonian metric space and let $T : X \rightarrow X$ is a non-Newtonian-β-ψ-φ-contractive mapping and satisfies the following conditions.

(i) T is a β-admissible

(ii) there exist $x_0 \in X$ such that $\beta(x_0, Tx_0) \dot{\geq} \dot{1}$;

(iii) T is digital continuous.

Then such u is a fixed point of T that is $Tu = u$.

Proof: Let $x_0 \in X$ such that $\beta(x_0, Tx_0) \dot{\geq} \dot{1}$ (such a point exist from the condition (ii)). Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then $u = x_{n_0}$ is a fixed point of T. So, we can assume that $x_n \neq x_{n+1}$ for all n. Since T is β-admissible, we have

$$\beta(x_0, x_1) = \beta(x_0, Tx_0) \dot{\geq} \dot{1} \quad \Longrightarrow \quad \beta(Tx_0, Tx_1) = \beta(x_1, x_2) \dot{\geq} \dot{1}.$$

Inductively, we have

$$\beta(x_n, x_{n+1}) \dot{\geq} \dot{1} \qquad \text{For all } n = 0,1,2 \dots \dots .. \quad (2)$$

From (1) and (2), it follows that for all $n \geq 1$, we have

$$\psi\big(d_N(x_n, x_{n+1})\big) = \psi(d_N(Tx_{n-1}, Tx_n))$$
$$\dot{\leq} \beta(x_{n-1}, x_n) \dot{\times} \psi(d_N(Tx_{n-1}, Tx_n))$$
$$\dot{\leq} \psi\big(d_N(x_{n-1}, x_n)\big) \dot{-} \varphi\big(d_N(x_{n-1}, x_n)\big) \qquad (3)$$

Hence, we have

$$\psi\big(d_N(x_n, x_{n+1})\big) \dot{\leq} \psi\,\big(d_N(x_{n-1}, x_n)\big) \dot{-} \varphi\big(d_N(x_{n-1}, x_n)\big). \qquad (4)$$

Consequently, the sequence $\{d_N(x_n, x_{n+1})\}$ is non-decreasing for all $n \in \mathbb{N}$.

Taking $n \to \infty$ in (4), and ψ and φ are continuous functions. So we have

$$\lim_{n\to\infty} d_N(x_n, x_{n+1}) = \dot{0} \qquad (5)$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. If it is not, then there exist $\varepsilon \dot{>} \dot{0}$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of sequence $\{x_n\}$ where n_k is the smallest index for which $n_k > m_k > k$ with$d_N\big(x_{m_k}, x_{n_k}\big) \dot{\geq} \varepsilon$. (6)

Then, $d_N\big(x_{m_k}, x_{n_k-1}\big) \dot{\leq} \varepsilon$. (7)

Using (6) and (7) we obtain

$$\varepsilon \dot{\leq} d_N\big(x_{m_k}, x_{n_k}\big) \dot{\leq} \big[d_N\big(x_{m_k}, x_{n_k-1}\big) \dot{+} d_N\big(x_{n_k-1}, x_{n_k}\big)\big] \dot{<} \varepsilon \dot{+} d_N\big(x_{n_k-1}, x_{n_k}\big). \qquad (8)$$

Taking the upper and lower limit as $k \to \infty$, we conclude

$$\varepsilon \dot{\leq} \lim_{k\to\infty} inf\, d_N\big(x_{m_k}, x_{n_k}\big) \dot{\leq} \lim_{k\to\infty} sup\, d_N\big(x_{m_k}, x_{n_k}\big) \dot{\leq} \varepsilon. \quad (9)$$

Using (9) and (5), we obtain

$$\varepsilon \dot{\leq} \lim_{k\to\infty} inf\; d_N\big(x_{m_k+1}, x_{n_k-1}\big). \qquad (10)$$

Moreover,

$$\varepsilon \dot{\leq} d_N\big(x_{m_k}, x_{n_k}\big) \dot{\leq} d_N\big(x_{m_k}, x_{m_k+1}\big) \dot{+} d_N\big(x_{m_k+1}, x_{n_k}\big) \qquad (11)$$

With taking the upper limit as $k \to \infty$ in (11) as follows

$$\varepsilon \dot{\leq} \lim_{k\to\infty} sup\, d_N\left(x_{m_k+1}, x_{n_k}\right) \quad (12)$$

By using (1), we have

$$\psi\left(d_N\left(x_{m_k+1}, x_{n_k}\right)\right) \dot{\leq} \beta\left(x_{m_k}, x_{n_k-1}\right) \dot{\times} \psi\left(d_N\left(Tx_{m_k}, Tx_{n_k-1}\right)\right),$$

$$\dot{\leq} \psi\left(d_N\left(x_{m_k}, x_{n_k-1}\right)\right) \dot{-} \varphi\left(d_N\left(x_{m_k}, x_{n_k-1}\right)\right). \quad (13)$$

Thus, from (12) and (13), we have

$$\psi(\varepsilon) \dot{\leq} \psi(\varepsilon) \dot{-} \varphi(\varepsilon) \quad (14)$$

which is not possible. Hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty}\{x_n\} = z$. (15)

Since T is continuous,

$$T(z) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = z$$

Hence, z is a fixed point of T.

If we replace the continuity condition (iii), Theorem 3.2 remains true. This statement is given as follows.

Theorem 3.3 Let (X, d_N) be a complete non-Newtonian metric space and let $T : X \to X$ is a non-Newtonian-β-ψ-φ-contractive mapping and satisfies the following conditions.

(i) T is a β-admissible

(ii) there exist $x_0 \in X$ such that $\beta(x_0, Tx_0) \dot{\geq} \dot{1}$;

(iii) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}) \dot{\geq} \dot{1}$ for all n and $x_n \overset{N}{\to} x \in X$ as $n \to \infty$, then there exists a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\beta\left(x_{n_k}, x\right) \dot{\geq} \dot{1}$ for all k.

Then, such z is a fixed point of T, that is $Tz = z$.

Proof: From proof of Theorem 3.2, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ is cauchy in (X, d, ρ) and converges to some $z \in X$. Consider (15),

$$\lim_{k\to\infty} d_N\left(x_{n_k+1}, Tz\right) = d_N(z, Tz) \quad (16)$$

Holds. By the assumption on X, we have

$$\psi\left(d_N\left(x_{n_k+1}, Tz\right)\right) \dot{\leq} \beta\left(x_{n_k}, z\right) \dot{\times} \psi\left(d_N\left(Tx_{n_k}, Tz\right)\right)$$

$$\dot{\leq} \psi\left(d_N\left(x_{n_k}, z\right)\right) \dot{-} \varphi\left(d_N\left(x_{n_k}, z\right)\right) \quad (17)$$

Since $\beta(x_n, z) \dot{\geq} \dot{1}$ we have

$$\psi\left(d_N(Tz, z)\right) \dot{\leq} \psi\left(d_N(Tz, Tx_n) \dot{+} d_N(Tx_n, z)\right)$$

$$\dot{\leq} \psi\big(d_N(Tz,Tx_n)\big)\dot{+}\psi\big(d_N(Tx_n,z)\big)$$
$$\dot{\leq} \beta(z,x_n)\dot{\times}\psi\big(d_N(Tz,Tx_n)\big)\dot{+}\psi\big(d_N(Tx_n,z)\big)$$
$$\dot{\leq} \psi\big(d_N(z,x_n)\big)\dot{-}\varphi\big(d_N(z,x_n)\big) \qquad (18)$$

Let $n \to \infty$ in (18), we have $\psi\big(d_N(Tz,z)\big) \leq 0$. Hence, z is a fixed point of T, or equivalently, $z = Tz$.

Corollary 3.4 Let (X, d_N) be a complete non-Newtonian metric space and $T: X \to X$ be such that

$$d_N(Tx,Ty) \dot{\leq} d_N(x,y)\dot{-}\varphi\big(d_N(x,y)\big)$$

for all $x, y \in X$. Then, T has a fixed point.

Proof: To prove this corollary it is suffices to take $\beta(x,y) = \dot{1}$ and $\psi(t) = t$ in theorem 3.2.

Theorem 3.5: Let (X, d_N) be a non- Newtonian metric space and let $T : X \to X$ is an nN-β-ψ-contractive mapping and satisfies the following conditions.

(i) T is a β-admissible

(ii) there exist $x_0 \in X$ such that $\beta(x_0, Tx_0) \dot{\geq} \dot{1}$;

(iii) T is nN-continuous for some $u \in X$, then $\beta(u,u) \dot{\geq} \dot{1}$.

(iv) $\beta(x,y)\dot{\times}\psi\big(d_N(Tx,Ty)\big) \dot{\leq} \psi\big(M(x,y)\big)\dot{-}\varphi\big(M(x,y)\big) \qquad (19)$

Where

$$M(x,y) = max\left\{d_N(x,y), d_N(x,Tx), d_N(y,Ty), \frac{d_N(x,Ty)\dot{+}d_N(y,Tx)}{\alpha(2)}\right\}; \text{ For all } x,y \in X. \quad (20)$$

Then such u is a fixed point of T that is $Tu = u$.

Proof: Let $x_0 \in X$ such that $\beta(x_0, Tx_0) \dot{\geq} \dot{1}$ (such a point exist from the condition (ii)). Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then $u = x_{n_0}$ is a fixed point of T. So, we can assume that $x_n \neq x_{n+1}$ for all n. Since T is β-admissible, we have

$$\beta(x_0,x_1) = \beta(x_0,Tx_0) \dot{\geq} \dot{1} \quad \Longrightarrow \quad \beta(Tx_0,Tx_1) = \beta(x_1,x_2) \dot{\geq} \dot{1}.$$

Inductively, we have

$$\beta(x_n,x_{n+1}) \dot{\geq} \dot{1} \quad \text{For all } n = 0,1,2 \ldots\ldots..(21)$$

From (19) and (21), it follows that for all $n \geq 1$, we have

$$\psi\big(d_N(x_n,x_{n+1})\big) = \psi(d_N(Tx_{n-1},Tx_n))$$
$$\dot{\leq} \beta(x_{n-1},Tx_n)\dot{\times}\psi(d_N(Tx_{n-1},Tx_n))$$
$$\dot{\leq} \psi\big(M(x_{n-1},x_n)\big)\dot{-}\varphi\big(M(x_{n-1},x_n)\big) \qquad (22)$$

Where

$$M(x_{n-1}, x_n) = max\left\{d_N(x_{n-1}, x_n), d_N(x_{n-1}, Tx_{n-1}), d_N(x_n, Tx_n), \frac{d_N(x_{n-1},Tx_n)\dot{+}d_N(x_n,Tx_{n-1})}{\alpha(2)}\right\}$$

$$= max\left\{d_N(x_{n-1}, x_n), d_N(x_n, x_{n+1}), \frac{d_N(x_{n-1},x_{n+1})\dot{+}d_N(x_n,x_n)}{\alpha(2)}\right\};$$

$$\dot{\leq} max\left\{d_N(x_{n-1}, x_n), d_N(x_n, x_{n+1}), \frac{d_N(x_{n-1},x_n)\dot{+}d_N(x_n,x_{n+1})}{\alpha(2)}\right\};$$

$$= max\left\{d_N(x_{n-1}, x_n), d_N(x_n, x_{n+1}), \frac{d_N(x_{n-1},x_n)\dot{+}d_N(x_n,x_{n+1})}{\alpha(2)}\right\}; \quad (23)$$

If for some n, $M(x_n, x_{n+1}) = d_N(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and (22) and (23) we obtain $\psi\big(d_N(x_n, x_{n+1})\big) \dot{\leq} \psi\ \big(d_N(x_n, x_{n+1})\big)\dot{-}\varphi\big(d_N(x_n, x_{n+1})\big) \dot{<} \psi\big(d_N(x_n, x_{n+1})\big)$. which is not possible.

Hence $M(x_n, x_{n+1}) = d_N(x_{n-1}, x_n)(\neq 0)$ then from (22) and (23) we have

$$\psi\big(d_N(x_n, x_{n+1})\big) \dot{\leq} \psi\ \big(d_N(x_{n-1}, x_n)\big)\dot{-}\varphi\big(d_N(x_{n-1}, x_n)\big). \quad (24)$$

Consequently, the sequence $\{d_N(x_n, x_{n+1})\}$ is non-decreasing for all $n \in \mathbb{N}$.

Taking $n \to \infty$ in (24), and ψ and φ are continuous functions. So we have

$$\lim_{n\to\infty} d_N(x_n, x_{n+1}) = 0. \quad (25)$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. If it is not, then there exist $\varepsilon \dot{>} \dot{0}$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of sequence $\{x_n\}$ where n_k is the smallest index for which $n_k > m_k > k$ with $d_N\big(x_{m_k}, x_{n_k}\big) \dot{\geq} \varepsilon$. (26)

Then, $d_N\big(x_{m_k}, x_{n_k-1}\big) \dot{\leq} \varepsilon$. (27)

Using (26) and (27) we obtain

$$\varepsilon \dot{\leq} d_N\big(x_{m_k}, x_{n_k}\big) \dot{\leq} \big[d_N\big(x_{m_k}, x_{n_k-1}\big)\dot{+}d_N\big(x_{n_k-1}, x_{n_k}\big)\big] \dot{<} \varepsilon\dot{+}d\big(x_{n_k-1}, x_{n_k}\big). \quad (28)$$

Taking the upper and lower limit as $k \to \infty$, we conclude

$$\varepsilon \dot{\leq} \lim_{k\to\infty} inf\, d_N\big(x_{m_k}, x_{n_k}\big) \dot{\leq} \lim_{k\to\infty} sup\, d_N\big(x_{m_k}, x_{n_k}\big) \dot{\leq} \varepsilon. \quad (29)$$

By the (iii) property of non-Newtonian metric space, we have

$$d_N\big(x_{m_{k+1}}, x_{n_k}\big) \dot{\leq} d_N\big(x_{m_{k+1}}, x_{m_k}\big)\dot{+}d_N\big(x_{m_k}, x_{n_k-1}\big)\dot{+}d_N\big(x_{n_{k-1}}, x_{n_k}\big), \quad (30)$$

with taking the upper limit as $k \to \infty$ in (30), we obtain

$$\lim_{k\to\infty} sup\, d_N\big(x_{m_{k+1}}, x_{n_k}\big) \dot{\leq} \varepsilon. \quad (31)$$

By the (iii) property of non-Newtonian metric space, we have

$$d_N\big(x_{m_k+1}, x_{n_k-1}\big) \dot{\leq} d_N\big(x_{m_k+1}, x_{m_k}\big)\dot{+}d_N\big(x_{m_k}, x_{n_k-1}\big), \quad (32)$$

by taking the upper limit as $k \to \infty$ in (32), we obtain

$$\lim_{k\to\infty} sup\, d_N\big(x_{m_{k+1}}, x_{n_k-1}\big) \dot{\leq} \varepsilon. \quad (33)$$

On the other hand,

$$d_N\left(x_{m_k}, x_{n_k}\right) \dot{\leq} d_N\left(x_{m_k}, x_{m_k+1}\right) \dot{+} d_N\left(x_{m_k+1}, x_{n_k-1}\right) \dot{+} d_N\left(x_{n_{k-1}}, x_{n_k}\right). \quad (34)$$

Using (29) and (25), we obtain

$$\varepsilon \dot{\leq} \lim_{k\to\infty} inf\, d_N\left(x_{m_k+1}, x_{n_k-1}\right). \quad (35)$$

Moreover,

$$\varepsilon \dot{\leq} d_N\left(x_{m_k}, x_{n_k}\right) \dot{\leq} d_N\left(x_{m_k}, x_{m_k+1}\right) \dot{+} d_N\left(x_{m_k+1}, x_{n_k}\right), \quad (36)$$

with taking the upper limit as $k \to \infty$ in (36) as follows

$$\varepsilon \dot{\leq} \lim_{k\to\infty} sup\, d_N\left(x_{m_{k+1}}, x_{n_k}\right) \quad (37)$$

By using (19), we have

$$\psi\left(d_N\left(x_{m_k+1}, x_{n_k}\right)\right) \dot{\leq} \beta\left(x_{m_k}, x_{n_k-1}\right) \dot{\times} \psi\left(d_N\left(Tx_{m_k}, Tx_{n_k-1}\right)\right),$$
$$\dot{\leq} \psi\left(M\left(x_{m_k}, x_{n_{k-1}}\right)\right) \dot{-} \varphi\left(M\left(x_{m_k}, x_{n_{k-1}}\right)\right). \quad (38)$$

Where

$$M\left(x_{m_k}, x_{n_{k-1}}\right) = max\left\{d_N\left(x_{m_k}, x_{n_{k-1}}\right), d_N\left(x_{m_k}, x_{m_{k+1}}\right), d_N\left(x_{n_{k-1}}, x_{n_k}\right), \frac{d_N\left(x_{m_k}, x_{n_k}\right) \dot{+} d_N\left(x_{n_{k-1}}, x_{m_{k+1}}\right)}{\alpha(2)}\right\} \quad (39)$$

On taking the upper limit as $k \to \infty$, from (25), (27), (29) and (33) we obtain

$$\lim_{k\to\infty} supM\left(x_{m_k}, x_{n_{k-1}}\right) = max\{\varepsilon, \dot{0}, \dot{0}, \varepsilon\} = \varepsilon. \quad (40)$$

Thus, from (19) and (20), we have

$$\psi(\varepsilon) \dot{\leq} \psi(\varepsilon) \dot{-} \varphi(\varepsilon) \quad (41)$$

which is not possible. Hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$, such that, $\lim_{n\to\infty}\{x_n\} = z$. (42)

Since T is continuous,

$$T(z) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = z$$

Hence, z is a fixed point of T.

If we replace the continuity condition (iii), Theorem 3.2 remains true. This statement is given as follows.

Theorem 3.6: Let (X, d_N) be a non- Newtonian metric space and let $T : X \to X$ is a β-ψ-φ-contractive mapping and satisfies the following conditions.

(iv) T is a β-admissible

(v) there exist $x_0 \in X$ such that $\beta(x_0, Tx_0) \dot{\geq} \dot{1}$;

(vi) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}) \dot{\geq} \dot{1}$ for all n and $x_n \overset{N}{\to} x \in X$ as $n \to \infty$, then there exists a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\beta\left(x_{n_k}, x\right) \dot{\geq} \dot{1}$ for all k

Then, such z is a fixed point of T, that is $Tz = z$.

Proof: From proof of Theorem 3.3, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ is cauchy in (X, d_N) and converges to some $z \in X$. Consider (42),

$$\lim_{k\to\infty} d_N\left(x_{n_k+1}, Tz\right) = d_N(z, Tz) \quad (43)$$

holds. By the assumption on X, we have

$$\psi\left(d_N\left(x_{n_{k+1}}, Tz\right)\right) \dot{\leq} \beta\left(x_{n_k}, z\right) \dot{\times} \psi\left(d_N\left(Tx_{n_k}, Tz\right)\right)$$
$$\dot{\leq} \psi\left(M\left(x_{n_k}, z\right)\right) \dot{-} \varphi\left(M\left(x_{n_k}, z\right)\right) \quad (44)$$

Where

$$M\left(x_{n_k}, z\right) = max\left\{d_N\left(x_{n_k}, z\right), d_N\left(x_{n_k}, Tx_{n_k}\right), d_N(z, Tz), \frac{d_N\left(x_{n_k}, Tz\right) \dot{+} d_N\left(z, Tx_{n_k}\right)}{\alpha(2)}\right\},$$
$$= max\left\{d_N\left(x_{n_k}, z\right), d_N\left(x_{n_k}, x_{n_{k+1}}\right), d_N(z, Tz), \frac{d_N\left(x_{n_k}, Tz\right) \dot{+} d_N\left(z, x_{n_{k+1}}\right)}{\alpha(2)}\right\}.$$

With (25) and (43), we have

$$\lim_{k\to\infty} M\left(x_{n_k}, z\right) = d_N(z, Tz). \quad (45)$$

Since $\beta(x_n, z) \dot{\geq} \dot{1}$ we have

$$\psi\left(d_N(Tz, z)\right) \dot{\leq} \psi\left(d_N(Tz, Tx_n) \dot{+} d_N(Tx_n, z)\right)$$
$$\dot{\leq} \psi\left(d_N(Tz, Tx_n)\right) \dot{+} \psi\left(d_N(Tx_n, z)\right)$$
$$\dot{\leq} \beta(z, x_n) \dot{\times} \psi\left(d_N(Tz, Tx_n)\right) \dot{+} \psi\left(d_N(Tx_n, z)\right)$$
$$\dot{\leq} \psi\left(M(z, x_n)\right) \dot{-} \varphi\left(M(z, x_n)\right) \quad (46)$$

Let $n \to \infty$ in (46), we have $\psi\left(d_N(Tz.z)\right) \dot{\leq} \dot{0}$. Hence z is a fixed point of T, or equivalently, $z = Tz$.

Corollary 3.7: Let (X, d_N) be a non- Newtonian metric space and $T: X \to X$ be such that

$$d_N(Tx, Ty) \dot{\leq} M(x, y) \dot{-} \varphi\left(M(x, y)\right)$$

for all $x, y \in X$ where $M(x, y)$ defined by (20). Then, T has a fixed point.

Proof: To prove this corollary it is suffices to take $\beta(x, y) = \dot{1}$ and $\psi(t) = t$ in theorem 3.5.

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