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The C-Transformation of the Riemann Zeta Function and the Riemann Hypothesis

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Abstract

Caceres derived the C-transformation of the Riemann zeta function. A variant of this function is used to derive results pertaining to the Riemann hypothesis.
Keywords Riemann zeta function, C-transformation, Riemann hypothesis

1. INTRODUCTION

Caceres [1] defined the C-transformation as

$$C_n\{f\} = \sum_{k=1}^{n} f(k) - \int f(n)dn \tag{1}$$

and derived the following function

$$Y(z,n) = n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+\beta^2]}[((1-\alpha)\cdot\cos(\beta\ln(n)) + \beta\cdot\sin(\beta\ln))) + \tag{2}$$

$$i(\beta\cdot\cos(\beta\ln(n)) - (1-\alpha)\cdot\sin(\beta\ln(n)))]. \tag{3}$$

Caceres' equation [24] is

$$Re(C_n\{f\}) = \sum_{k=1}^{n} k^{-\alpha}(\cos(\beta\cdot\ln(k)) + \tag{4}$$

$$\frac{1}{[(1-\alpha)^2+\beta^2]}(n^{(1-\alpha)}[(1-\alpha)\cdot\cos(\beta\cdot\ln(n)) + \beta\cdot\sin(\beta\cdot\ln(n))]) \tag{5}$$

where $f = \frac{1}{n^z}$. Caceres' equation [25] is

$$Im(C_n\{f\}) = -\sum_{k=1}^{n} k^{-\alpha}(\sin(\beta\cdot\ln(k)) + \tag{6}$$

$$\frac{1}{[(1-\alpha)^2+\beta^2]}(n^{(1-\alpha)}[\beta\cdot\cos(\beta\cdot\ln(n))-(1-\alpha)\cdot\sin(\beta\cdot\ln(n))]) \tag{7}$$

Let $\zeta'(z,n)$ denote $\sum_{k=1}^{n}\frac{1}{(n-k+1)^z}$. $C_n\{\frac{1}{n-k+1}\}$ is derived from these formulas by substituting $\sum_{k=1}^{n}(n-k+1)^{-\alpha}$ for $\sum_{k=1}^{n}k^{-\alpha}$.

A plot of $C_n\{\frac{1}{(n-k+1)^z}\}-\zeta'(z,n)$ for the first non-trivial zeta function zero and $n\leq 10000$ is

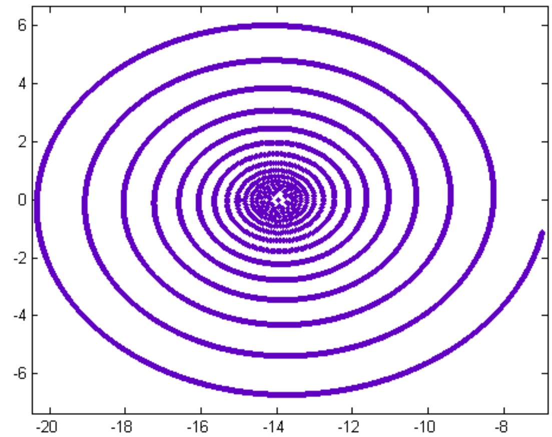

Figure 1

A plot of the inflection points (where the y-component of the curve approaches zero from above and then becomes negative) is

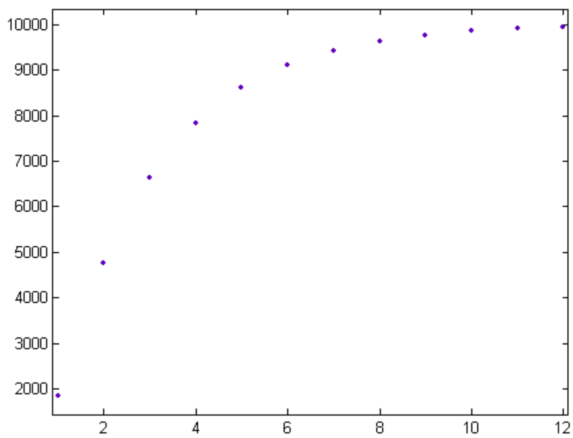

Figure 2

The n values of the inflection points are 1857, 4772, 6642, 7842, 8613, 9108, 9426, 9630, 9761, 9846, 9900, and 9935.

Let $Y'(z,n)$ denote

$$(n-k+1)^{(1-\alpha)}\frac{1}{[(1-\alpha)^2+\beta^2]}[((1-\alpha)\cdot\cos(\beta\ln(n))+\beta\cdot\sin(\beta\ln))+ \tag{8}$$

$$i(\beta\cdot\cos(\beta\ln(n))-(1-\alpha)\cdot\sin(\beta\ln(n)))] \tag{9}$$

where $k=1$ to n.

A plot of $\zeta'(z,n)-Y'(z,n)$ for the first zeta function zero and $n\leq 100000$ is

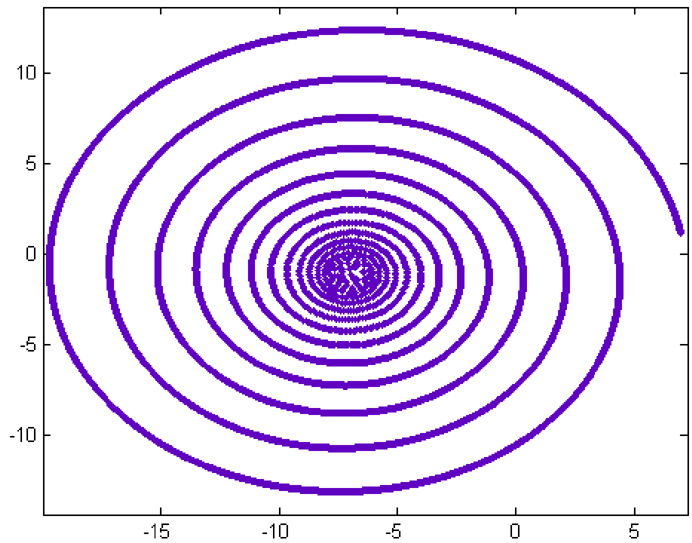

Figure 3

The n values of the inflection points are 1857, 4772, 6642, 7843, 8613, 9108, 9426, 9630, 9761, 9846, 9900, and 9935. Even though the spirals rotate in different directions, the n values of the inflection points are the same as those for $C_n\{\frac{1}{(n-k+1)^z}\}-\zeta'(z,n)$ (except for the fourth n value which is one less).

A plot of $\zeta'(z,n)-Y'(z,n)-(C_n\{\frac{1}{(n-k+1)^z}\}-\zeta'(z,n))$ and $C_n\{\frac{1}{(n-k+1)^z}\}-\zeta'(z,n)$ versus n is

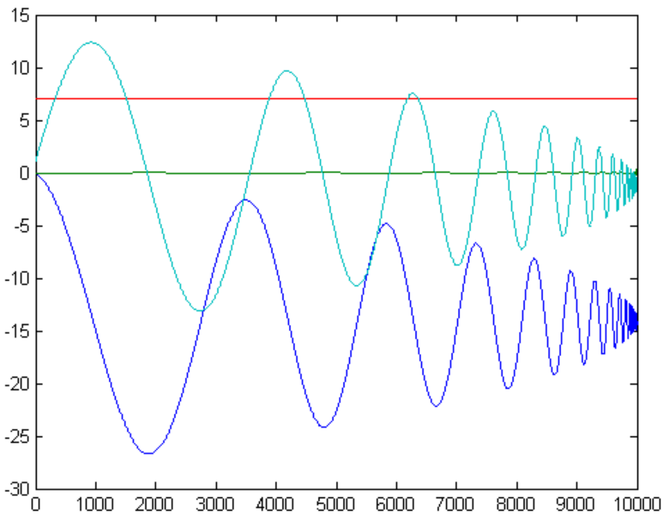

Figure 4

The bottom-most curve is the real components of $\zeta'(z,n) - Y'(z,n) - (C_n\{\frac{1}{(n-k+1)^z}\} - \zeta'(z,n))$ and the curve starting with 0 is the imaginary components. A plot of this curve is

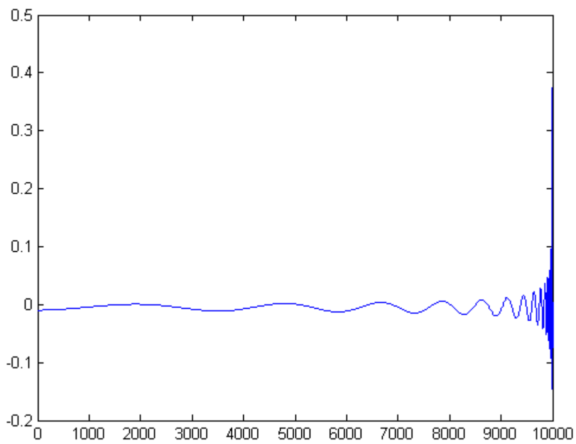

Figure 5

A plot of the last ten points of the curve is

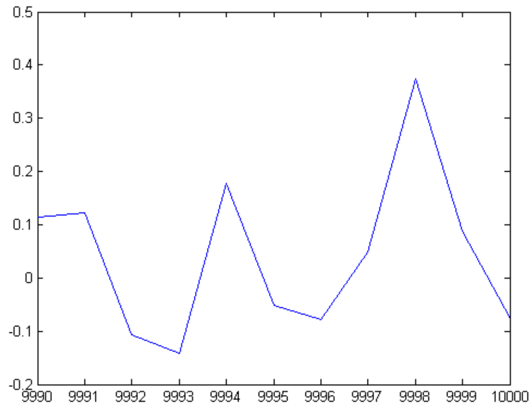

Figure 6

As the upper bound of the n values gets larger, the last point slowly approaches 0 but the rest of the curve oscillates as above. For $n \leq 10000$, the last value is -0.0756. For $n \leq 100000$, 200000, and 300000, the respective values are -0.0716, -0.0704, and -0.0700.

The real component of $C_n\{\frac{1}{(n-k+1)^z}\} - \zeta'(z,n)$ is the straight line. A plot of the imaginary components of $C_n\{\frac{1}{(n-k+1)^z}\} - \zeta'(z,n)$ and the real components of $\zeta'(z,n) - Y'(z,n) - (C_n\{\frac{1}{(n-k+1)^z}\} - \zeta'(z,n))$ is

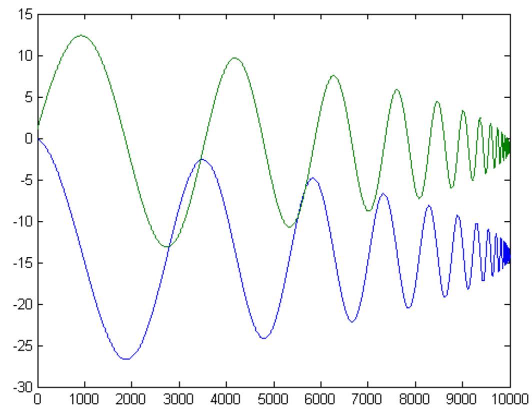

Figure 7

Other than being out of phase, the two curves are almost equal. The n value of the inflection point half-way down an oscillation of the upper curve is almost equal to the n value of the inflection point at the bottom of an oscillation of the lower curve.

2. ABSOLUTE SQUARES

A "polynomial" form of $Y(z,n)$ (derived by Caceres) is

$$|Y(z,n)|^2 \geq n^{2(1-a)}/[(1-\alpha)^2 + \beta^2] \tag{10}$$

A plot $|Y(z,n)|^2$ and the real and imaginary components of the right-hand side of the inequality for the first zeta function zero and $n \leq 1000$ is

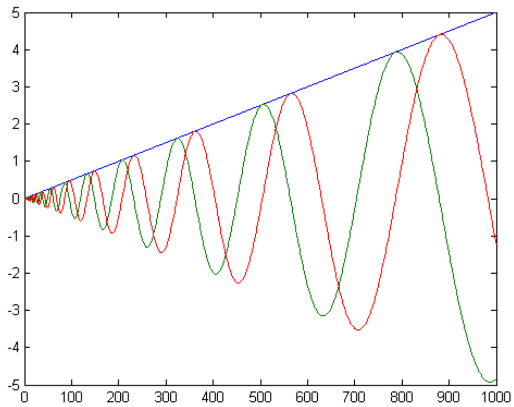

Figure 8

The slope of $|Y(z,n)|^2$ is

$$d(|Y(z,n)|^2)/dn = 2(1-\alpha)n^{1-2\alpha}/[(1-\alpha)^2 + \beta^2] \tag{11}$$

A plot $|Y(z,n)|^2$ and the real and imaginary components of the right-hand side of the inequality for $z = (0.5, 34.0)$ is

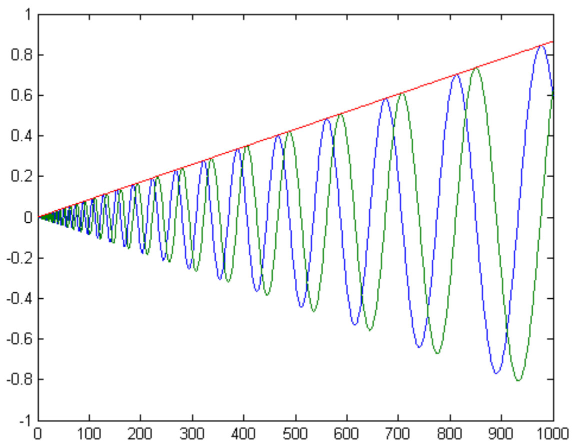

Figure 9

The slopes of the two above lines are 0.004999 and 0.0008649 (relative to n). The ratio of these two slopes is about 5.78. The ratio of $(1-0.5)^2+34.0^2$ and $(1-0.5)^2+14.134725^2$ is about 5.78.

Caceres [2] proved the following

Theorem 1. *$|Y(z,n)|^2$ is a straight line if and only if $\alpha = 1/2$.*

A "polynomial" form of $Y'(z,n)$ is

$$|Y'(z,n)|^2 \geq (n-k+1)^{2(1-a)}/[(1-\alpha)^2+\beta^2] \tag{12}$$

for $k = 1$ to n. A plot $|Y'(z,n)|^2$ and the real and imaginary components of the right-hand side of the inequality for the first zeta function zero and $n \leq 10000$ is

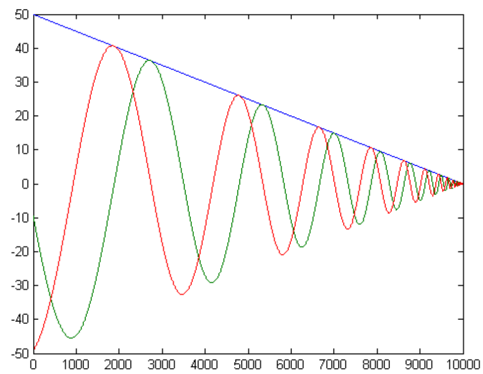

Figure 10

A plot of $|C_n\{\frac{1}{(n-k+1)^z}\} - \zeta'(z,n)|^2$ versus n for the first zeta function zero and $n \leq 10000$ is

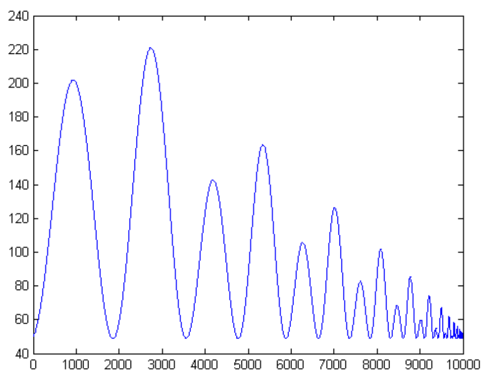

Figure 11

The portion of the curve with linearly decreasing peaks corresponds to $\zeta'(z,n)$.

A plot of $|\zeta'(z,n)|^2 - |Y'(z,n)|^2 - |C_n\{\frac{1}{(n-k+1)^z}\} - \zeta'(z,n)|^2$ for the first zeta function zero and $n \leq 10000$ is

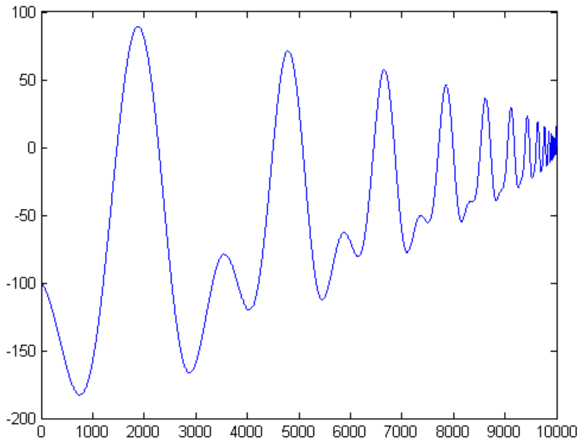

Figure 12

A plot of the last ten points is

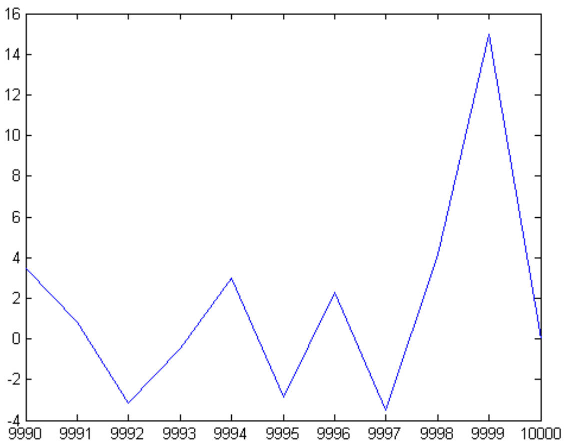

Figure 13

For larger upper bounds of n, the last value approaches 0. For $n \leq 10000$, 20000, 30000,. . ., 100000, the last values are 0.0187, -0.0327, 0.0406, -0.0696, -0.0001, 0, 0.0001, -0.0001, 0.0001, and -0.0001 respectively.

A plot of $|\zeta'(z,n)|^2 - |Y'(z,n)|^2 - |C_n\{\frac{1}{(n-k+1)^z}\} - \zeta'(z,n)|^2$ for $z = (0.5, 10.0)$ and $n \leq 10000$ is

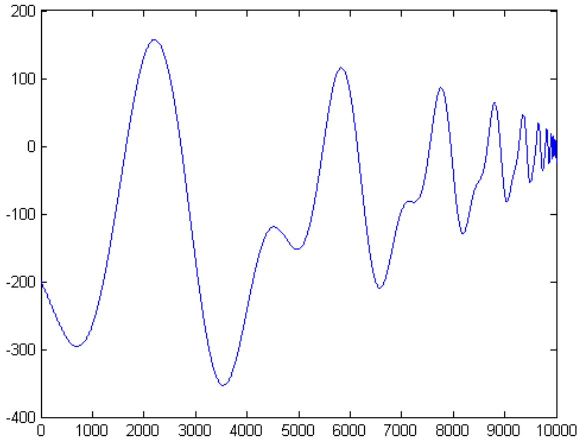

Figure 14

A plot of the last ten values is

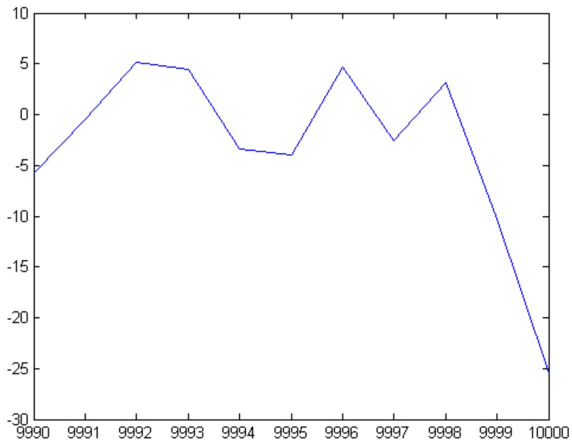

Figure 15

The last value distinguishes this z value from the zeta function zeros. For $n \leq 100000$, the last value is 83.8037.

A plot of $|\zeta'(z,n)|^2 - |Y'(z,n)|^2 - |C_n\{\frac{1}{(n-k+1)^z}\} - \zeta'(z,n)|^2$ for $z = (0.4, 14.13472514173470)$ (the first zeta function zero is $z = (0.5, 14.13472514173470)$) and $n \leq 10000$ is

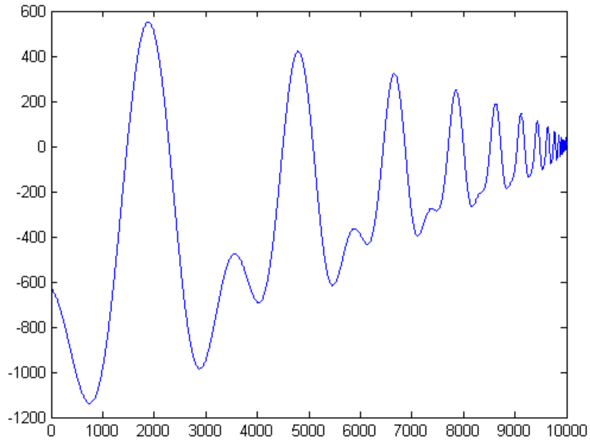

Figure 16

A plot of the last ten values is

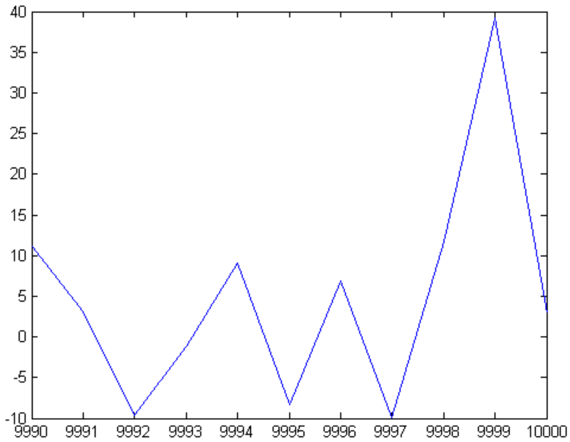

Figure 17

The last value is 2.9402. This is much larger than the corresponding value for the first zeta function zero. For $n \leq 100000$, the last value is 7.3380.

For the first ten zeta function zeros and $n \leq 200000$, the last values are 0, 0, 0, 0, 0, 0.0053, -0.0030, 0.0189, -0.0026, and 0.0125 respectively. In general, the last value appears to converge only if z is a zeta function zero. This is relevant to the Riemann hypothesis since $\lim_{n\to\infty} |\zeta(z,n)|^2$ appears to equal $\lim_{n\to\infty} |\zeta'(z,n)|^2$ for zeta function zeros. A plot of $|\zeta(z,n)|^2$ and $|\zeta'(z,n)|^2$ for the first zeta function zero and $n \leq 10000$ is

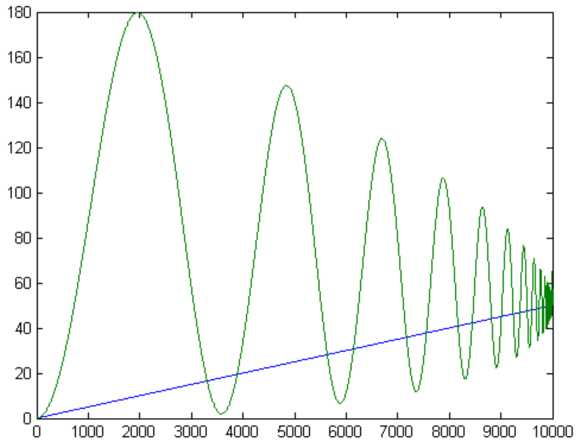

Figure 18

3. METHODS

The C code for computing $\zeta(z,n)$, $Y(z,n)$ and $C_n\{\frac{1}{n^z}\}$ is as follows.

```
#include <math.h>
#include <stdio.h>
//
// zeta function (when noinc=0)
// X(z,n) (when noinc=1)
//
unsigned int max=1000;
double s=0.50;
//double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
```

```
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
unsigned int xmin=0; // usually set to 0
unsigned int noinc=0; // if set, don't add additional term
unsigned int out=1; // set to 1 when not finding inflection points
void main() {
unsigned int x;
double temp1,temps,tempt,sumr,sumi,a,b,olds,oldt;
FILE *Outfp;
Outfp = fopen("spiral1.dat","w");
sumr=0.0;
sumi=0.0;
olds=0.0;
oldt=0.0;
for (x=1; x<=max; x++) {
    if (s>=0.0)
        temp1=pow((double)x,s);
    else {
        temp1=pow((double)x,-s);
        temp1=1.0/temp1;
        }
    a=temp1*(cos(t*log(x)));
    b=temp1*(sin(t*log(x)));
    temp1=a*a+b*b;
    sumr=sumr+a/temp1;
    sumi=sumi-b/temp1;
    if (s>=0.0)
        temp1=pow((double)max,s);
    else {
        temp1=pow((double)max,-s);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(max)));
    tempt=temp1*(sin(t*log(max)));
    if (noinc==0) {
        temps=sumr+temps/2.0;
        tempt=sumi+temps/2.0;
```

```
        }
    else {
        temps=sumr;
        tempt=sumi;
        }
    if (x>xmin) {
        if (out==1)
            fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
        if ((out==2)&&((olds>0.0)&&(temps<0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==3)&&((oldt>0.0)&&(tempt<0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==4)&&((olds<0.0)&&(temps>0.0)&&(x>1)))
            fprintf(Outfp,"                                    if
((out==5)&&((oldt<0.0)&&(tempt>0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        olds=temps;
        oldt=tempt;
        }
    }
printf(" %.10llf %.10llf \n",temps,tempt);
fclose(Outfp);
return;
}

#include <math.h>
#include <stdio.h>
//
// Y(z,n)
//
unsigned int maxn=1000;
unsigned int out=1;
double s=0.50;
//double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
double t=30.42487612585951;
```

```
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
void main() {
double temp1,temps,tempt,x,olds,oldt;
unsigned int n,max;
FILE *Outfp;
Outfp = fopen("spiral2.dat","w");
olds=0.0;
oldt=0.0;
for (n=1; n<=maxn; n++) {
    max=n;
    x=1.0-s;
    if (x>=0.0)
        temp1=pow((double)max,x);
    else {
        temp1=pow((double)max,-x);
        temp1=1.0/temp1;
        }
    temps=temp1*(x*cos(t*log(max))+t*sin(t*log(max)));
    temps=temps/(x*x+t*t);
    tempt=temp1*(t*cos(t*log(max))-x*sin(t*log(max)));
    tempt=tempt/(x*x+t*t);
    if (out==1)
        fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
    if ((out==2)&&((olds>0.0)&&(temps<0.0)&&(n>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",n,temps,tempt);
    if ((out==3)&&((oldt>0.0)&&(tempt<0.0)&&(n>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",n,temps,tempt);
    if ((out==4)&&((olds<0.0)&&(temps>0.0)&&(n>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",n,temps,tempt);
    if ((out==5)&&((oldt<0.0)&&(tempt>0.0)&&(n>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",n,temps,tempt);
    olds=temps;
    oldt=tempt;
```

```
    }
printf(” %.10llf %.10llf \n”,temps,tempt);
fclose(Outfp);
return;
}

#include <math.h>
#include <stdio.h>
//
// C-transformation
//
unsigned int max=1000;
double s=0.50;
double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
unsigned int out=1; // use out=3 otherwise
void main() {
unsigned int x,z;
double temp1,temps,tempt,sumr,sumi,a,b,olds,oldt,y;
FILE *Outfp;
Outfp = fopen(”ctrans.dat”,”w”);
sumr=0.0;
sumi=0.0;
olds=0.0;
oldt=0.0;
for (x=1; x<=max; x++) {
    z=max;
    if (s>=0.0)
        temp1=pow((double)z,s);
```

```
    else {
        temp1=pow((double)z,-s);
        temp1=1.0/temp1;
        }
    a=temp1*(cos(t*log(z)));
    b=temp1*(sin(t*log(z)));
    temp1=a*a+b*b;
    sumr=sumr+a/temp1;
    sumi=sumi-b/temp1;
    y=1-s;
    if (y>=0.0)
        temp1=pow((double)max,y);
    else {
        temp1=pow((double)max,-y);
        temp1=1.0/temp1;
        }
    temps=temp1*(y*cos(t*log(max))+t*sin(t*log(max)));
    tempt=temp1*(t*cos(t*log(max))-y*sin(t*log(max)));
    temps=temps/(y*y+t*t);
    tempt=tempt/(y*y+t*t);
    temps=sumr+temps;
    tempt=tempt-sumi;
    if (out==1)
        fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
    if ((out==2)&&((olds>0.0)&&(temps<0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==3)&&((oldt>0.0)&&(tempt<0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==4)&&((olds<0.0)&&(temps>0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==5)&&((oldt<0.0)&&(tempt>0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    olds=temps;
    oldt=tempt;
    }
printf(" %.10llf %.10llf \n",temps,tempt);
fclose(Outfp);
return; }
```

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[2] Caceres, Pedro, Proof of the Riemann Hypothesis using the decomposition Zeta(z)=X(z)-Y(z), RG/339841648, 2020