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[j, k]-Set Domination in Cycle Graph

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Abstract

Domination is an important theoretical concept in Graph theory. By [j, k]-Set Domination a subset $D \subseteq V$ in a Graph G = (V, E) is a [j, k]-Set if every vertex $v \in V - D,\ j \leq |\ N(v) \cap D\ | \leq k$ for j = 1 and k = 2 that is every vertex $v \in V - D$ is just adjacent to at least j but not more than k vertices in D. The Domination number of G is denoted by $\gamma_{[j,k]}(G)$, which the minimum cardinality of dominating set G. In this paper Generalization of [j, k]-Domination number of Cycle Graph has been studied.

Subject Classification 05C69

Keywords: Dominating set, Domination number, [j, k]-Dominating set, [j, k]-Domination number, Cycle Graph.

INTRODUCTION

The basic concept and introduction of Graph theory have been learned from D.B. West [7]. Fundamentals of Domination in Graphs were introduced by T.W. Haynes et al [3]. Middle and central of various graphs have been explained in [1, 2]-Domination in Graphs by Murugesan. N and Deepa. S. Nair [4]. [1, 2]-Sets of various graphs have been understood and explained by M.Chellali et al [1]. [1, 2]- Domination in Generalized Petersen Graphs was clearly explained by Chengye Zhao and Chao Zhang [2]. The relation between domination, [1, 2]-Domination, and bounds for different Graphs have been extracted from Xiaozing Yang, Baoyindureng Wu [8]. The concept of [1, k]-Domination in Graphs has been explained by E. Sampath Kumar [6]. Several types of domination have been explained in efficient domination by Robert R. Rubalcaba and Peter J. Slater [5].

PRELIMINARY

Let G = (V, E) be a simple connected undirected finite Graph. The order and size of the vertex V is $n = |V| and\ m = |E|$. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ of vertices adjacent to v. The closed neighborhood of the vertex $v \in V$ is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is deg $(v) = |N(v)|$. The minimum and maximum degree of Graph G is denoted by $\delta(G)$ and $\Delta(G)$. That is $\delta(G) = \min_{v \in V(G)} \{d(v)\}$ and $\Delta(G) = \max_{v \in V(G)} \{d(v)\}$

Definition 2.1 A set $D \in V$ of vertices in a Graph G= (V, E) is a Dominating set if and only if every vertex $v \in V - D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of Dominating set in D.

Definition 2.2 Let G = (V, E) be a Graph, and a set $D \subseteq V$ is a [j, k]-set if $j \leq |N(v) \cap D| \leq k$ for every vertex $v \in V - D$.

Definition 2.3 A Dominating set D of Graph G is called efficient if $|N(v) \cap D| = 1$ for every vertex $v \in V - D$. That is a Dominating set D is efficient if and only if every vertex is Dominated exactly once.

Definition2.4 For a Graph G = (V, E) a set $D \subseteq V$ is independent if no two vertices in D are adjacent. The domination number is denoted by i (G).

Definition 2.5 A cycle is a Graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.

Definition 2.6: A subset V_D of V is a dominant component if there is a $v \in V_D$, such that v is adjacent to each vertex u in V_D.

Lemma 2.7: If γ (G) = r, then the dominant components in G are also r.

Lemma 2.8: V (G) = $\bigcup_{i} V_{Di}$

Lemma 2.9: if V (G) = $\bigcup_{i}^{r} V_{Di}$ and V_{Di}'s are disjoint then G contains a dominating set of size r.

Lemma 2.10: If the dominating components in G are disjoint then the corresponding dominating set is an [1, 1]- dominating set.

Definition 2.11: A set of vertices taken in order is said to be uniformly spaced if every vertex is at an equal distance from the preceding and succeeding vertices (if any)

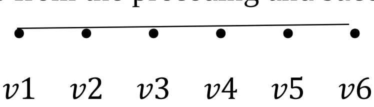

Figure 1.1: A graph in which $\{v_1, v_3, v_5\}$ are uniformly spaced.

Lemma 2.12: If D is a minimum dominating set with no two vertices adjacent then D is a [1, 1]-dominating set

Lemma 2.13:

In cycle C_n if $D = \{u, v, w, x, y, z\}$ is a minimum dominating set in which the vertices are uniformly spaced, then the sets $D + 1, D + 2$ are also minimum dominating sets.

Definition 2.14: In a cycle C_n, if $D = \{v_i, v_j, v_k\}$ a subset of $V(C_n)$, then D+r represents the subset of $\{v_{i+r,} v_{j+r}, v_{k+r}\}$, where

$$\boldsymbol{v_{l+r}} = \begin{cases} \boldsymbol{l+r \ \ if \ \ l+r \leq n} \\ \boldsymbol{l+r(modn) \ \ if \ l+r > n} \end{cases}$$

3. [1, 1]-Domination of the cycle graphs

Theorem 3.1: The number of [1, 1] - dominating sets of the cycles are

$$|\ D_{[1,1]}(C_n)| = \begin{cases} 3, & when\, n = 3t, \quad t = 1,2,3 \ldots \ldots \\ 3t+1, & when\, n = 3t+1, t = 1,2,3 \ldots \ldots \\ 0, & when\, n = 3t+2, t = 1,2,3 \ldots \ldots \end{cases}$$

Proof: Let $v_1, v_2, v_3 \cdots \cdots v_n$ are the vertices of the cycle C_n taken in order

i.e., $v_i \sim v_{i+1}$, i =1, 2, 3$\cdots n-1$ $and\ v_n \sim v_{0.}$ $since\ d(v_i) = 2$, each vertex $v_i\ in\ C_n$ dominates only two of its adjacent vertices. Hence the vertex v_i and its two neighbours v_{i-1}, v_{i+1} constitute a dominating component of three vertices. Thus, clearly, the set of all such $v_i's$ are minimum dominating set. This dominating set contains none of its neighbors in C_n $when\, n = 3t$, Hence $\gamma(C_n) = t\ when\, n = 3t,\ t = 1, \ 2, \ 3\cdots$. However when $n = 3t + 1\ or\ 3t + 2,$ the set of vertices dominate all vertices except 1 or 2 respectively. i.e., if the set of $v_i's$ is taken as $V_D = \{v_1, v_{4,} v_7, \ldots v_{3t-2}\}$ then V_D is minimum dominating set in $C_n\ when\, n = 3t$ and the vertices $v_{3t}\ in\ C_{3t+1}$ and $v_{3t}, v_{3t+1}\ in\ C_{3t+2}$ have no adjacent vertices in V_D. Hence $V_D \cup \{V_{3t}\}$ is a dominating set in C_{3t+1} $and\ either\ V_D\ \cup\ \{v_{3t}\}$ or $V_D \cup\ \{v_{3t+1}\}$ is a dominating set in C_{3t+2}. Since V_D is a minimum dominating set, the above two sets are also minimum dominating sets in the respective graph C_n. Therefore, to prove this theorem we consider three cases accordingly.

Case (i): $Let\ n = 3t, t = 1, 2, 3\cdots\cdots\cdots$

Here, it can be seen that the dominant components are $\{v_{3t,} v_{1,} v_2\}$, $\{v_3, v_4, v_5\} \cdots \cdots \{v_{3t-3}, v_{3t-2}, v_{3t-1}\}$ There are t number of dominant components in C_n, when n = 3t and $V_{D=}\{v_1, v_4, v_{7\ldots\ldots} v_{3t-5}, v_{3t-2}\}$ is a minimum dominating set. Since the above dominant components in V_D are disjoint it is an [1, 1]- dominating set. Also, each vertex in V_D is uniformly spaced in $V_{D,}$ the set $V_{D1=}\{v_2, v_5, v_8 \cdots \cdots v_{3t-1}\}$ and

$V_{D2} = \{v_3, v_6, v_9 \cdots \cdots v_{3t}\}$ are also [1,1] - dominating sets in $C_{n\,.}$

Hence $\gamma\ (C_n)\ =$ t and $\left| D_{[1,1]}(C_n) \right| = 3,$ when n = 3t.

Generalizing the above argument, it can be written as

$$D_{[1,1]}(C_n) = \begin{cases} \{v_r \mid r = 3k + s - 3 \\ k = 1, 2, 3 \cdots n \\ s\ is\ a\ fixed\ integer\} \\ \text{where s } = \text{ 1,2,3.} \end{cases}$$

Thus, $|D_{[1,1]}(C_n)| = 3, when\ n = 3t$

Case (ii) : $Let\ n = 3t + 1,\ t = 1,2,3 \cdots$

The sets $\{v_{3t+1}, v_1, v_2\}, \{v_3, v_4, v_5\} \cdots \cdots \{v_{3t-3}, v_{3t-2}, v_{3t-1}\}, \{v_{3t}\}$ form the role of dominating compounds. In each component, the vertices are written in the same order as placed in the cycle. Hence, the vertices written at the second place in each component together with the vertex v_{3t} form a minimum dominating set. Hence the minimum dominating number is merely the number of dominating components. This number is mearly nothing but the number of elements in the A.P 1, 4, 7….3t-2 plus one.

Hence the number of components = $\frac{3t-2-1}{3} + 1 + 1 = \frac{3t-3}{3} + 2 = t - 1 + 2 = t + 1$

It can be noted that the middle vertex is adjacent to every other two vertices in each of the above dominating components. Hence the set consisting of middle vertices together with v_{3t+1} forms a minimum dominating set, in which the vertices are uniformly spaced. Hence the set consisting of the vertices is uniformly spaced. Hence the set consisting of the vertices adjacent to $v_1, v_4, v_7 \cdots v_{3t-2}$ is also a minimum dominating set. Therefore it is interesting to see the number of such dominating sets. The following is the list of such dominating sets.

1. $\{v_1, v_4, v_7, \cdots v_{3t-2}, v_{3t+1}\}$
2. $\{v_2, v_5, v_8, \cdots v_{3t-1}, v_1\}$
3. $\{v_3, v_6, v_9, \cdots v_{3t}, v_2\}$
4. $\{v_4, v_7, v_{10}, \cdots v_{3t+1}, v_3\}$

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This process will end once the list contains the list of vertices is the same as in the first one. We can observe from the above list that the list contains a different set of vertices at each count and the count will end once we get the list as $\{v_{3t+1}, v_3, v_6 \cdots \cdots v_{3t-3}, v_{3t}\}$. So, the number of such lists is 3t+1. By combining the above arguments, we can generalize the result as follows: $for\ n = 1,2,3 \cdots$

$$D_{[1,1]}(\boldsymbol{c_{3t+1}})\,] = \begin{cases} \{v_r \mid \mathrm{r} = 3\mathrm{k} + \mathrm{s} - 3, & \text{If } 3\mathrm{k} + \mathrm{s} - 3 \leq 3\mathrm{n} + 1 \\ s \text{ is a fixed integer}\} & \\ \{v_q \mid 3\mathrm{k} + \mathrm{s} - 3 = \mathrm{p}\,(3\mathrm{n} + 1) + \mathrm{q}, & \text{If } 3\mathrm{k+s-3} > 3\mathrm{n+1} \\ \text{For some integer } p & \\ \text{s is a fixed integer}\} & \\ \text{where } k = 1,2,3\cdots(n+1); & \text{Where } s = 1,2,3\ldots(3n+1) \end{cases}$$

$Thus\ |D_{[1,1]}\ (c_{3t+1})| = 3t + 1$ and $\gamma_{[1,1]}(C_{3t+1}) = t + 1$

Case (iii): Let n=3t+2, t=1, 2, 3……

$$\{v_{3t+2}, v_1, v_2\}, \{\, v_3\ , v_4, v_5\}, \{v_6, v_7, v_8\} \ldots\ldots \{\, v_{3t+1}\ , v_{3t}\}$$

are dominant components of the cycle C_{3t+2}. There are (t+1) number of such components, they are disjoint V_D ={ v_1,v_4,v_7, …… v_{3t+1}} are uniformly dominating sets. So V_D is [1, 1]- dominating sets.

But $\{v_{3t+1}, v_1\}$ are not uniformly dominations set.

So C_{3t+2} is not [1, 1]- dominations set

$Hence\ D_{[1,1]}(C_{3t+2}) = \emptyset$

[1, 2]-Domination of cycle graph

Theorem 3.2: The number of [1, 2] dominating sets of Cycle C_n is

$$D_{[1,2]}(C_n) = \begin{cases} 3t + 1, when\ n = 3t + 1, & t = 1 \\ 3t + 1, when\ n = 3t + 1, & t = 2,3 \ldots\ldots \\ 3t + 2, when\ n = 3t + 2, & t = 1,2,3 \ldots\ldots \\ 0, when\ n = 3t, & t = 1,2,3 \ldots\ldots \end{cases}$$

Proof: let $v_1, v_2, v_3 \ldots\ldots v_n$ are the vertices of the cycle C_n taken in order ie $v_i \sim v_{i+1}$

I=1, 2, 3… (n-1) and v_n~v_1. Since d (v_i) = 2, each verter v_i in C_n dominate only two of its adjacent vertices. Hence the verter v_i and its two neighbors v_{i-1}, v_{i+1} constitute a dominating components of three vertices. Thus clearly the set of all such v_i's is a minimum dominating set.

Case (1): $let\ n = 3t + 1,\ t = 1$

Two dominating sets have been found. There are $\{v_1, v_3\}\ and\ \{v_2, v_4\}$

Case (2) Let n= 3t+1, t=2, 3, 4……

The sets $\{v_{3t+1}, v_t\}, \{v_2, v_3, v_4\}, \{v_5, v_6\}, \{v_7, v_8, v_9\}\{v_{10}, v_{11}, v_{12}\} \cdots\cdots \{v_{3t-2}, v_{3t-1}, v_{3t}\}$

Form the role of dominating components. In each component, the vertices are written in the same order as placed in the cycle. Hence the vertices written at the second place in each component together with the vertices $\{v_{3t-2}, v_{3t-1}, v_{3t}\}$ form a minimum dominating set. Hence the minimum dominating number is merely the number of dominating components. This number is nothing but the number of elements in the AP is 1, 3, 5 and 8, 11, 14 …… 3t-1.

Hence $\gamma_{[1,2]}\,(c_{3t+1}) = \frac{5-1}{2} + 1 + \frac{3t-1-8}{3} + 1 = 3 + \frac{3t-9}{3} + 1 = \frac{9+3t-9+3}{3} = \frac{3(1+t)}{3} = 1 + t$

It can be noted that the middle vertex is adjacent to every other two vertices in each of the above dominating components. Hence the set consisting of middle vertices together with v_{3t-1} forms a minimum dominating set in which the vertices are uniformly spaced.

The dominating sets of the cycle c_{3t+1} are

$\{v_1, v_3, v_5, v_8, v_{11} \dots\dots\dots v_{3t-7}, v_{3t-4}, v_{3t-1}\}$

$\{v_2, v_4, v_6, v_9, v_{12} \dots\dots\dots v_{3t-6}, v_{3t-3}, v_{3t}\}$

$\{v_3, v_5, v_7, v_{10}, v_{13} \dots\dots\dots v_{3t-5}, v_{3t-2}, v_{3t+1}\}$

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$\{v_{3t+1}, v_2, v_4 \dots\dots\dots v_{3t-8}, v_{3t-5}, v_{3t-2}\}$, 3t − k > 0

The above $v_D\,'s$ are the minimum dominating set. The components in $v_D\,'s$ are disjoint. This forms uniformly Dominating set except for three vertices. In three vertices v_i, v_{i+1}, v_{i+2}, $v_{i+1}$is dominated byv_i *and* v_{i+2}. So this forms two dominations, and other vertices form one domination. So the above dominating set forms double [1, 2] dominating sets.

Hence $|D_{[1,2]}(c_{3t+1}|$= 3t+1 and $\gamma(c_{3t+1}) = t + 1$

So the generalized above argument

1. $D_{[1,2]}(C_{3t+1}) =$

When n = 1

$\{v_r | r = 2k + s - 2, \quad If\ 2k + s - 2 \leq 3n + 1$

$k = 1,2$

$s\ is\ a\ fixed\ integer\}$

$where\ s = 1,2$

$When\ n = 2$

$\{v_r \mid r = 2k + s - 2 \quad if\ 2k + s - 2 \leq 3n + 1$

$K = 1,2,3$

$S\ is\ a\ fixed\ integer\}$

$Where\ s = 1, 2 \dots . (3n + 1)$

$\{v_q | 2k + s - 2 = p\,(3n + 1) + q, \quad If\ 2k + s - 2 > 3n + 1$

$for\ some\ integer\ pk = 1, 2, 3 \cdots (n+1)$

$s\ is\ a\ fixed\ integer\}$

$where\ s = 1, 2, 3 \cdots (3n+1)$

$When\ n = 3, 4, 5 \cdots$

$\{v_{r1}, v_{r2} \mid r1 = 2k1 + s1 - 2, \quad if \quad 2k1 + s1 - 2 \leq 3n + 1$

$r2 = 3k_2 + s_2 - 5 \quad if\ 3k2 + s2 - 5 \leq 3n + 1$

$k1 = 1, 2, 3$

$k2 = 4, 5, 6 \cdots \cdots (n+1)$

$s1, s2\ are\ fixed\ integer\}$

$where\ s1, s2 = 1, 2 \cdots \cdots (3n+1)$

$\{\ 2k + s1 - 2 = p(3n+1) + q1\ ,\ if\ 2k + s2 - 2 > 3n + 1$

$3k2 + s2 - 5 = p(3n+1) + q2, if\ 3k2 + s2 - 5 > 3n + 1$

For some integers p

k1=1, 2, 3

$k2 = 4, 5, 6 \ldots \ldots (n+1)$

$s1, s2$ Are fixed integer}

Where $s1, s2$ = 1, 2 …. (3n+1)

Case (ii) $let\ n = 3t + 2,\ \ t = 1, 2, 3, 4 \ldots \ldots \ldots$

The sets $\{v_{3t+2}, v_1\}$, $\{v_2, v_3, v_4\}$, $\{ v_5, v_6, v_7\}$, $\{v_8, v_{9,} v_{10}\} \cdots \cdots$ $\{v_{3t-1}, v_{3t}, v_{3t+1}\}$

Form the role of dominating components. In each component, the vertices are written in the same order as placed in the cycle. Hence the vertices written at the second place in each component together with the vertices $\{v_{3t-1}, v_{3t}, v_{3t+1}\}$ form a minimum dominating set. Hence the minimum dominating number is mearly the number of dominating components. This number is nothing but the number of elements in the AP is 1 and 3, 6, 9 …… 3t.

$$\gamma_{[1,2]}(C_{3t+2}) = 1 + \frac{3t-3}{3} + 1$$

$$= 3 + \frac{3t - 3 + 3}{3}$$

$$= \frac{3t + 3}{3} = 1 + t$$

It can be noted that the middle vertex is adjacent to every other two vertices in each of the above dominating components. Hence the set consisting of middle vertices together with v_{3t} forms a minimum dominating set in which the vertices are uniformly spaced.

The number of dominating components of the cycle C_{3t+2} is (t+1)

The dominating components of the cycle C_{3t+2} are disjoint

The dominating sets of the cycle C_{3t+2} are

$$\{v_1, v_3, v_6, v_9 \ldots\ldots v_{3t-1}, v_{3t-3}, v_{3t}\}$$

$$\{v_2, v_4, v_7, v_{10} \ldots\ldots v_{3t-5}, v_{3t-2}, v_{3t+1}\}$$

$$\{v_3, v_5, v_8, v_{11} \ldots\ldots v_{3t-4}, v_{3t-1}, v_{3t+2}\}$$

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$$\{v_{3t+2}, v_2, v_5, \ldots\ldots v_{3t-7}, v_{3t-4}, v_{3t-1}\} \qquad 3t-k>0$$

The above v_D's are minimum dominating sets. The components in $v_D's$ are disjoint. This forms uniformly dominating sets except two vertices. In three vertices v_i, v_{i+1}, v_{i+2}, v_{i+1} is dominated by the two vertices v_i and v_{i+2}. This forms two dominations other vertices form one domination. So the above dominating set forms [1, 2]-dominating set.

Hence $|D_{[1,2]}(c_{3t+2})|$ = 3t+2 and γ (c_{3t+2}) = t+1

So the generalizing the above argument we get

$ii. D_{[1,2]}(C_{3t+2})$ =

$when\ n = 1$

$\{v_r \mid r = 2k + s - 2 \qquad If\ 2k + s - 2 \leq 3n + 2$

K=1, 2, 3 ……. (n+1)

$s\ is\ a\ fixed\ integer\}$

$s = 1, 2 \cdots (3n + 2)$

$\{v_q \mid 2k + s - 2 = p(3n + 2) + q \qquad if\ 2k + s - 2 > 3n + 2$

$K = 1,2,3 \ldots.(n + 1)$

$S\ is\ a\ fixed\ integer\}$

$S = 1, 2 \cdots (3n + 2)$

$when\ n = 2, 3, 4 \cdots$

$\{v_{r1}, v_{r2} \mid r_1 = 2k1 + s1 - 2 \qquad if\ 2k1 + s1 - 2 \leq 3n + 2$

$r_2 = 3k2 + s2 - 4 \qquad if\ 3k2 + s2 - 4 \leq 3n + 2$

$k1 = 1, 2$

$k2 = 3, 4 \cdots (n + 2)$

$s1, s2\ are\ fixed\ integer$

$S1, s2\ =\ 1, 2, 3 \cdots\ (3n + 2)$

$\{v_{q1,} v_{q2} | 2k1 + s1 - 2 = p(3n + 2) + q1 \qquad if\ 2k1 + s1 - 2 > 3n + 2$

$3k2 + s2 - 4 = p(3n + 2) + q2 \qquad if\ 3k2 + s2 - 4 > 3n + 2$

For fixed p

K1 = 1, 2

K2 = 3, 4, 5…n+2

s1, s2 are fixed integer

s1, s2=1, 2, 3……. (3n+2)

Case (iii) Let n = 3t, t= 1, 2, 3

[1, 2] – dominating for the cycle C_n does not exist.

Contrary, let us assume that [1, 2] - domination exists for the cycleC_n.

We can construct the dominating components such as $\{v_2, v_3, v_4\}$, $\{v_5, v_6, v_7\}$………… $\{v_{3t}, v_1\}$

But vertex v_{3t-1} is not included in the sets, it is isolated.

In [1,1]- dominating set, the dominating components set contains three vertices in each set. But, here the last set contains only two vertices.

This brings us to the conclusion that our assumption is wrong.

So [1, 2]-domination does not exist for the cycle C_n when n=3t, t=1, 2, 3 ………..

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CONCLUSION

In this paper, we have generalized the [j, k]-Domination number of the cycle Graph. Similarly, we can study [j, k]-Domination number of some other special Graphs.

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