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A Gamma Function Pertaining to the Riemann Hypothesis

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Abstract

An analogue of Caceres' X(z)-Y(z) function is derived using the gamma function.
Keywords: C-transformation of Riemann zeta function, Riemann hypothesis

1. INTRODUCTION

Caceres [1] defined the C-transformation as

$$C_n\{f\} = \sum_{k=1}^{n} f(k) - \int f(n)dn \tag{1}$$

and derived the following function

$$X(z,n) = \sum_{k=1}^{n} k^{-\alpha}(\cos(\beta \cdot \ln(k)) + \frac{1}{2}n^{-\alpha}\cos(\beta\ln(n))+ \tag{2}$$

$$i(\sum_{k=1}^{n} k^{-\alpha}(\sin(\beta \cdot \ln(k)) + \frac{1}{2}n^{-\alpha}\sin(\beta\ln(n))). \tag{3}$$

The associated function is

$$Y(z,n) = n^{(1-\alpha)}\frac{1}{[(1-\alpha)^2+\beta^2]}[((1-\alpha)\cdot\cos(\beta\ln(n)) + \beta\cdot\sin(\beta\ln)))+ \tag{4}$$

$$i(\beta\cdot\cos(\beta\ln(n)) - (1-\alpha)\cdot\sin(\beta\ln(n)))]. \tag{5}$$

Let $\zeta(z,n)$ denote $\sum_1^n \frac{1}{n^z}$. A plot of $C_n\{\frac{1}{n^z}\}-\zeta(z,n)$ and $\zeta(z,n)$ for the first non-trivial zeta function zero and $n \leq 1000$ is

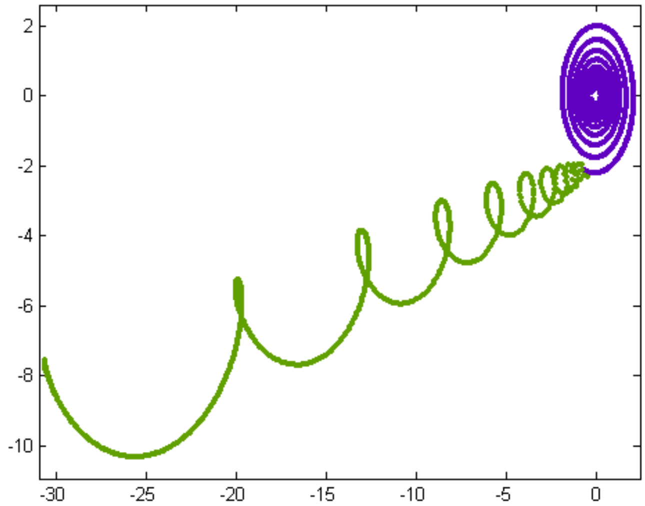

Figure 1

Let $G(z,n)$ denote

$$\frac{\Gamma_n(-\frac{\alpha}{2})\zeta_n(z)}{2\pi} \tag{6}$$

where α is the real component of z and Γ_n denotes the gamma function. A plot of this function for the first zeta function zero is

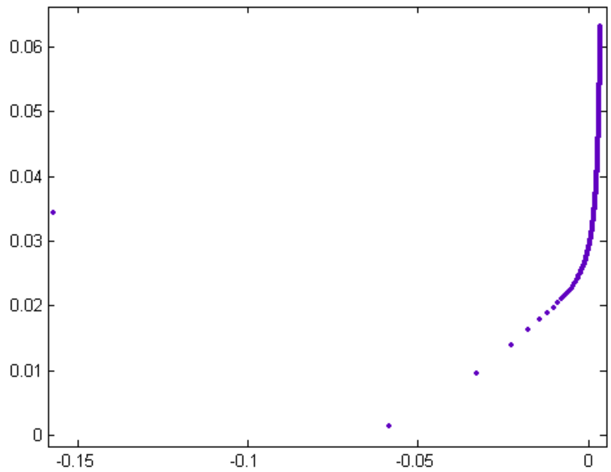

Figure 2

$G(z, n)$ is a line (not necessarily straight) only if z is a zeta function zero.

A plot of $(C_n\{\frac{1}{n^z}\} - \zeta(z, n))G(z, n)$ for the first zeta function zero and $n \leq 1000$ is

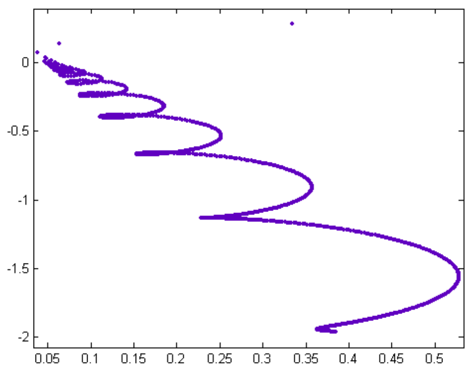

Figure 3

In general, mutiplying a logarithmic spiral by $G(z, n)$ reduces its size and changes its orientation.

2. THE SECOND ZETA FUNCTION ZERO

A plot of $X(z, n)$ and $X(z, n)G(z, n)$ for the second zeta function zero and $n \leq 1000$ is

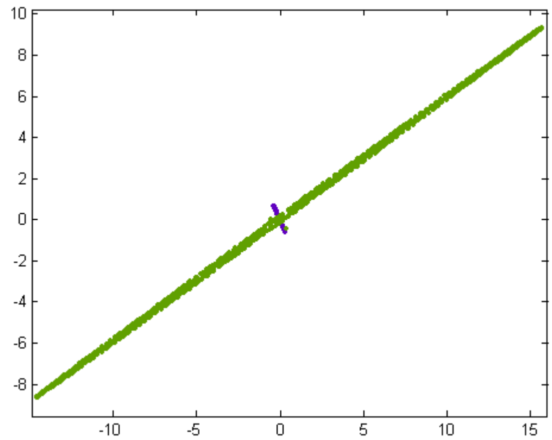

Figure 4

Both curves are edge-on logarithmic spirals. The n values of the inflection points of $X(z, n)G(z, n)$ (where the curve approaches the x-axis and then increases) are 4, 7, 9, 12, 16, 22, 29, 39, 53, 71, 96, 129, 174, 235, 317, 427, 576, and 776. The n values of the inflection points of $X(z, n)$ are 6, 8, 11, 14, 19, 25, 34, 46, 62, 83, 112, 151, 203, 273, 368, 496, 669, and 903. The n values of the inflection points of $\zeta(z, n)$ are 6, 8, 11, 15, 20, 27, 36, 49, 66, 88, 119, 161, 217, 292, 394, 531, 716, and 966. Each transformation reduces the slope of the logarithms of the n values of the inflection points.

A plot of the n values of the inflection points of $X(z, n)$ versus the n values of the inflection points of $X(z, n)G(z, n)$ is

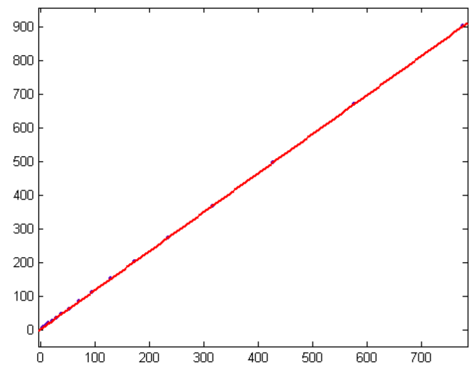

Figure 5

For a linear least-squares fit of the curve, $p_1 = 1.161$ with a 95% confidence interval of (1.16, 1.162), $p_2 = 0.4159$ with a 95% confidence interval of (0.1004, 0.7313), SSE=4.009, R-squared=1, and RMSE=0.5006.

A plot of the n values of the inflection points of $\zeta(z, n)$ versus the n values of the inflection points of $X(z, n)G(z, n)$ is

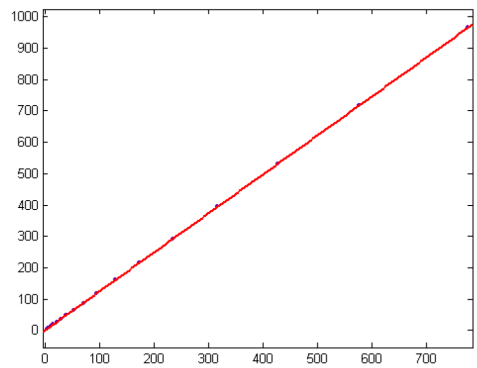

Figure 6

For a linear least-squares fit of the curve, $p_1 = 1.244$ with a 95% confidence interval of (1.243, 1.245), $p_2 = -0.01129$ with a 95% confidence interval of (-0.3198, 0.2972), SSE=3.834, R-squared=1, and RMSE=0.4895.

A plot of $X(z,n)G(z,n) - Y(z,n)$ is

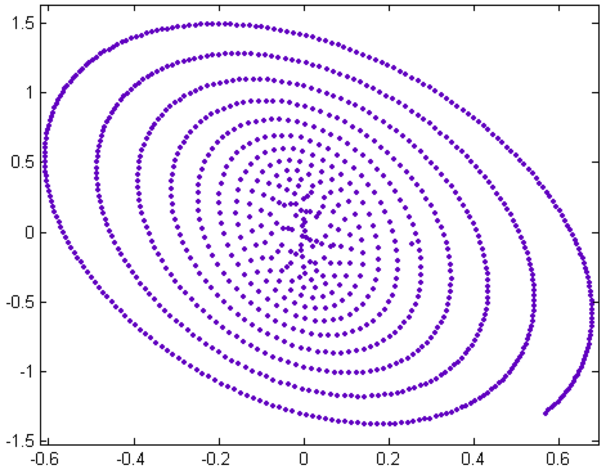

Figure 7

This is a 3-dimensional logarithmic spiral. The n values of the inflection points are 3, 7, 10, 13, 17, 23, 31, 42, 57, 76, 102, 138, 186, 250, 337, 454, 611, and 824. A plot of the n values of the inflection points versus the n values of the inflection points of $X(z,n)G(z,n)$ is

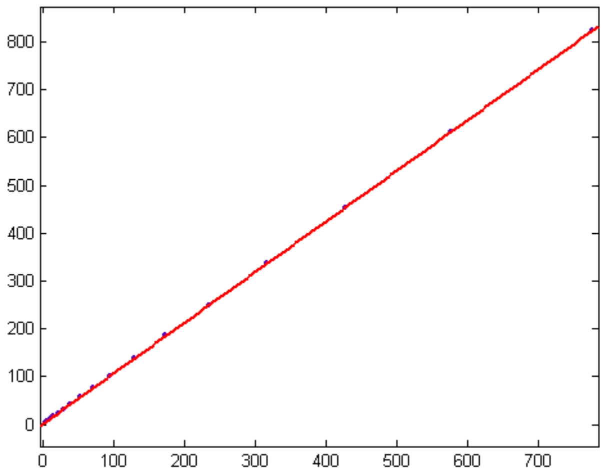

Figure 8

For a linear least-squares fit of the curve, $p_1 = 1.062$ with a 95% confidence interval of (1.06, 1.063), $p_2 = 0.2333$ with a 95% confidence interval of (-0.1601, 0.6268), SSE=6.237, R-squared=1, and RMSE=0.6243.

3. COMPLEX VALUES HAVING THE IMAGINARY COMPONENT OF A ZETA FUNCTION ZERO

A plot of $X(z, n)G(z, n)$ for $z = (0.40, 32.93506158773919)$ and $n \leq 10000$ is

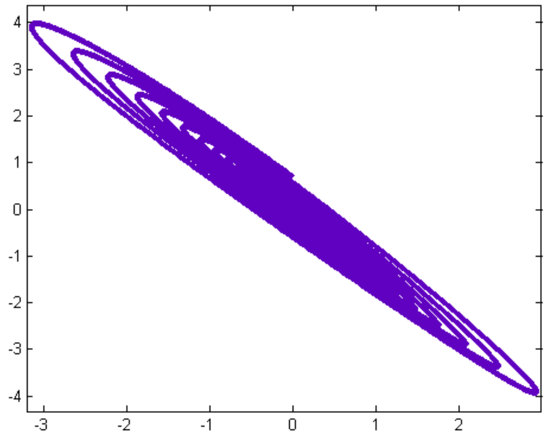

Figure 9

The fifth zeta function zero is (0.50, 32.93506158773919). A plot of the logarithms of the n values of the inflection points is

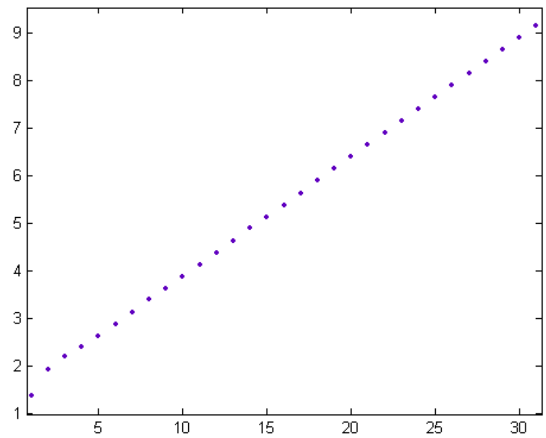

Figure 10

For a linear least-squares fit of the curve (disregarding the first point), $p_1 = 0.2497$ with a 95% confidence interval of (0.2491, 0.2503), $p_2 = 1.404$ with a 95% confidence interval of (1.393, 1.415), SSE=0.005214, R-squared=1, and RMSE=0.01365.

A plot of the logarithms of the n values of the inflection points of $X(z,n)G(z,n)$ versus the logarithms of the n values of the first 31 inflection points of $X(z,n)$ (out of a total of 37) is

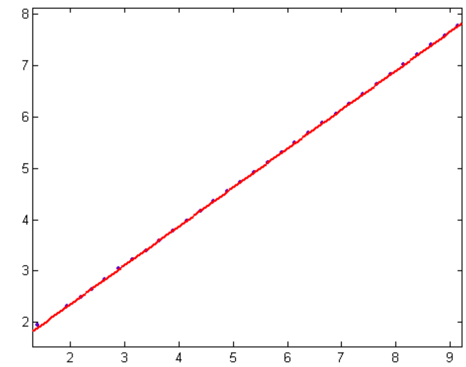

Figure 11

For a linear least-squares fit of the curve, $p_1 = 0.7583$ with a 95% confidence interval of (0.7561, 0.7605), $p_2 = 0.8354$ with a 95% confidence interval of (0.8228, 0.8481), SSE=0.005108, R-squared=0.9999, and RMSE=0.01327.

4. THE THIRD ZETA FUNCTION ZERO

A plot of $Y(z,n)$ for the third zeta function zero and $n \leq 1000$ is

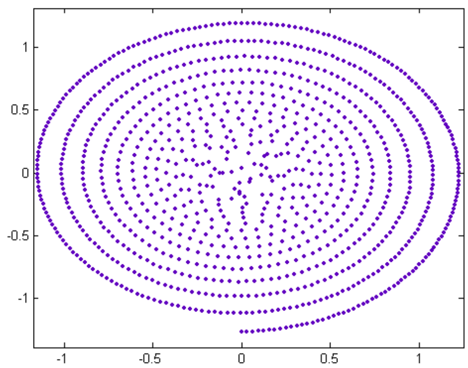

Figure 12

A plot of the logarithms of the n values of the inflection points is

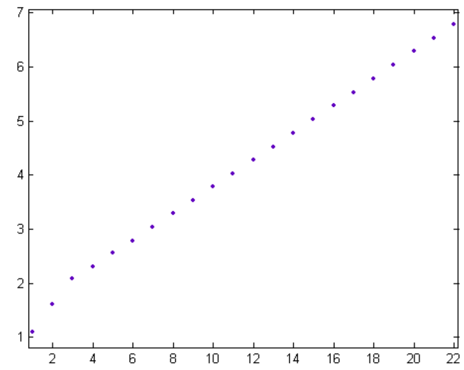

Figure 13

For a linear least-squares fit of the curve (disregarding the first two points), $p_1 = 0.2485$ with a 95% confidence interval of (0.2475, 0.2495, $p_2 = 1.305$ with a 95% confidence interval of (1.291, 1.318), SSE=0.002735, R-squared=0.9999, and RMSE=0.01233. For a linear least-squares fit of the logarithms of the n values of the inflection points of the third zeta function zero (disregarding the first point), $p_1 = 0.251$ with a 95% confidence interval of (0.2498, 0.2522), $p_2 = 1.761$ with a 95% confidence interval of (1.746, 1.776), SSE=0.003166, R-squared=0.9999, and RMSE=0.01365.

A plot of $X(z, n)G(z, n) - Y(z, n)$ for $n \leq 10000$ is

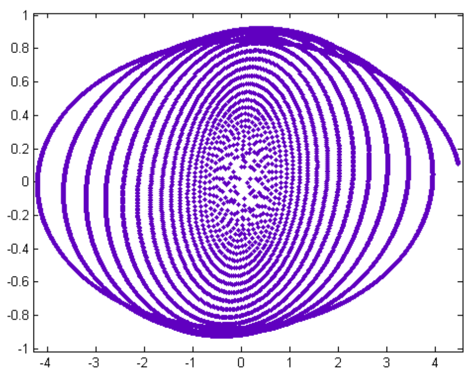

Figure 14

This is a 3-dimensional logarithmic spiral. A plot of the logarithms of the n values of the inflection points is

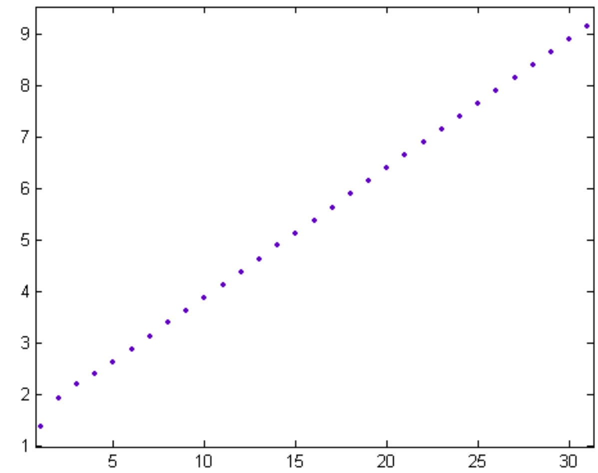

Figure 15

For a linear least-squares fit of the curve, (disregarding the first point), $p_1 = 0.2497$ with a 95% confidence interval of (0.2491, 0.2503), $p_2 = 1.404$ with a 95% confidence interval of (1.393, 1.415), SSE=0.005214, R-squared=1, and RMSE=0.01365.

5. THE SIXTH ZETA FUNCTION ZERO

A plot of $X(z,n) - Y(z,n)$ for the sixth zeta function zero and $n \leq 10000$ is

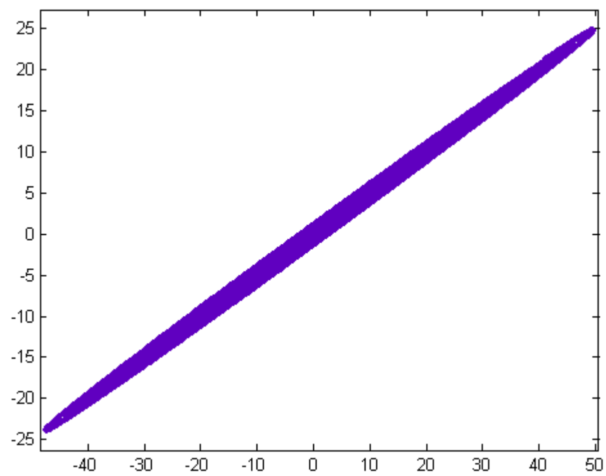

Figure 16

This is an edge-on logarithmic spiral. A plot of the logarithms of the n values of the inflection points is

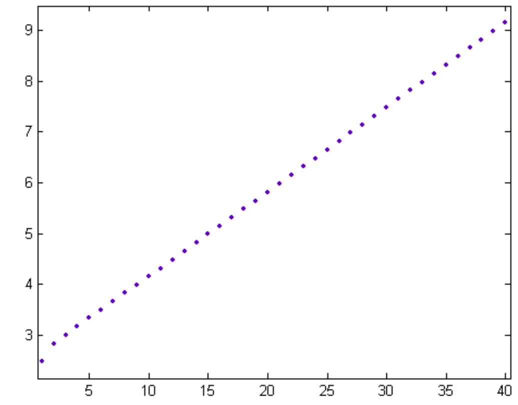

Figure 17

For a linear least-squares fit of the curve (disregarding the first point), $p_1 = 0.1663$ with a 95% confidence interval of (0.1661, 0.1665), $p_2 = 2.493$ with a 95% confidence interval of (2.489, 2.495), SSE=0.001404, R-squared=1, and RMSE=0.00616.

6. SQUARES OF ABSOLUTE VALUES

A "polynomial" form of $Y(z, n)$ (derived by Caceres) is

$$|Y_p(z, n)|^2 = n^{2(1-a)}/[(1 - \alpha)^2 + \beta^2] \quad (7)$$

A plot of $|X(z, n)G(z, n) - Y(z, n)|$, $|X(z, n)G(z, n)|$, $|Y(z, n)|^2$, and the real and imaginary components of $|Y_p(z, n)|^2$ for the fourth zeta function zero and $n \leq 100000$ is

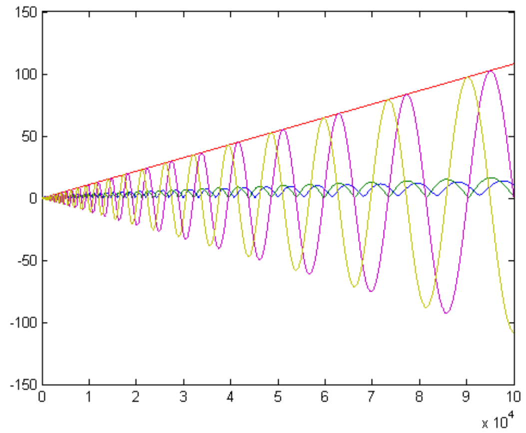

Figure 18

$|Y(z,n)|^2$ is the straight line that touches the tops of the oscillations of the real and imaginary components of $|Y_p(z,n)|^2$.

A plot of $|X(z,n)G(z,n) - Y(z,n)|$ and the real and imaginary components of $\zeta(z,n)$ is

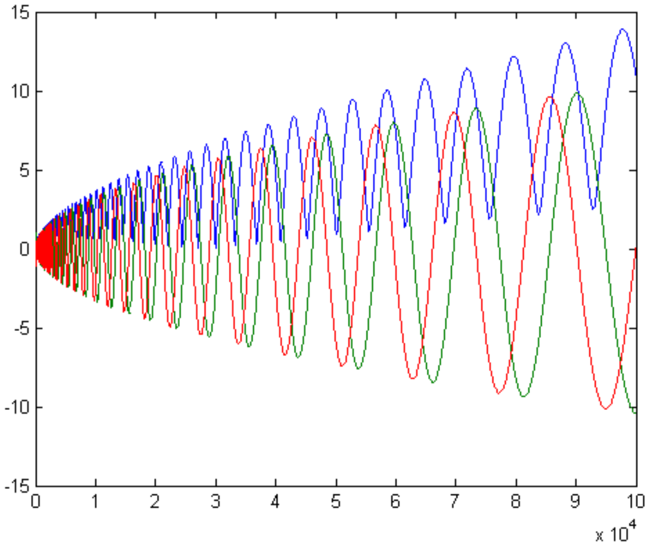

Figure 19

A plot of the logarithms of the n values of the inflection points of $\zeta(z,n)$ is

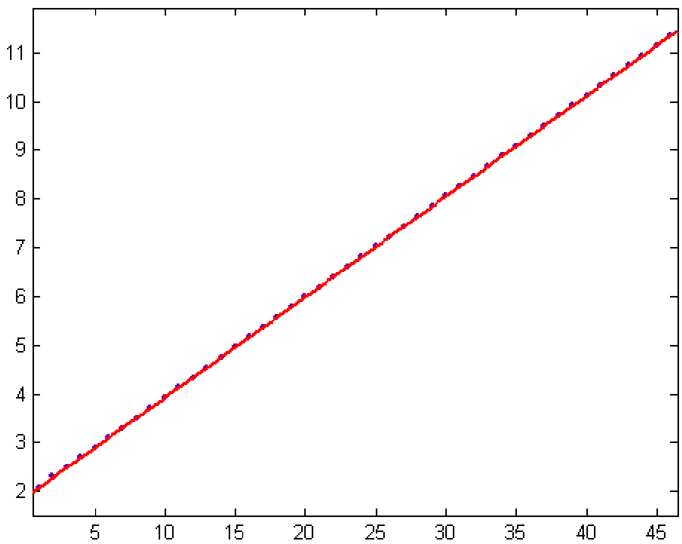

Figure 20

For a linear least-squares fit of the curve, $p_1 = 0.2064$ with a 95% confidence interval of (0.2062, 0.2065), $p_2 = 1.863$ with a 95% confidence interval of (1.859, 1.867), SSE=0.00202, R-squared=1, and RMSE=0.006776.

A plot of the logarithms of the n values of the inflection points of $|X(z,n)G(z,n) - Y(z,n)|$ is

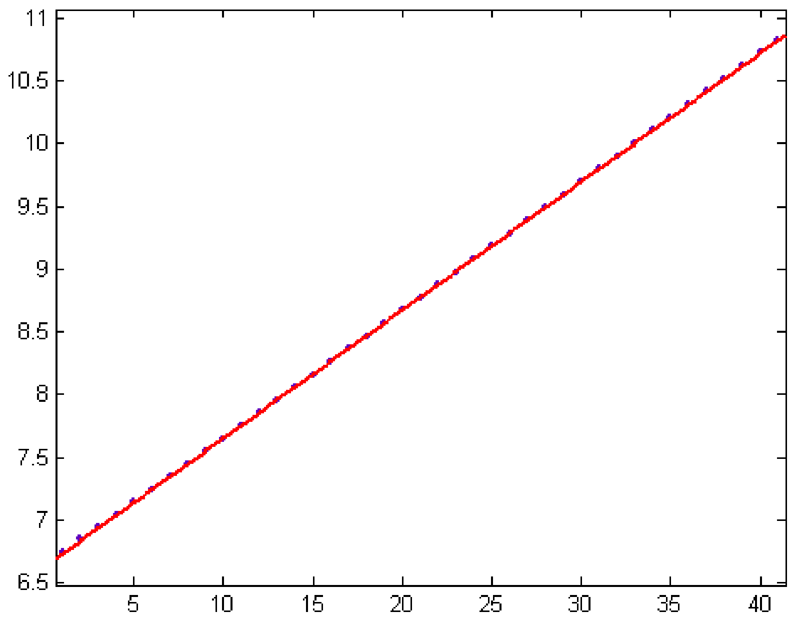

Figure 21

For a linear least-squares fit of the curve, $p_1 = 0.1023$ with a 95% confidence interval of (0.1021, 0.1024), $p_2 = 6.629$ with a 95% confidence interval of (6.626, 6.632), SSE=0.0009739, R-squared=1, and RMSE=0.004997.

There is then a linear relationship between the logarithms of the n values of the inflection points of the two curves.

7. CONCLUSION

Let $X(z)$ denote $\lim_{n\to\infty} X(z,n)$ and let $Y(z)$ denote $\lim_{n\to\infty} Y(z,n)$. Caceres [2] proved that $\zeta(z) = X(z) - Y(z)$ and used this to prove the Riemann hypothesis. The above may be of some value as an alternate approach.

8. METHODS

The C code for computing the function involving the gamma function, $X(z,n)$, $Y(z,n)$, and the C-transformation is as follows.

```
#include <math.h>
#include <stdio.h>
unsigned int max=1000;
double s=0.50; // s/2 is used
double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
double pi=3.14159265359;
unsigned int xmin=0;
void main() {
unsigned int temp,x;
double temp1,temps,tempt,prods,a,b,c,d,e,f,sums,sumt;
FILE *Outfp;
Outfp = fopen(”funct4am.dat”,”w”);
f=2.0*pi;
temp1=f*f;
e=0.0;
f=-f/temp1;
prods=1.0;
sums=0.0;
sumt=0.0;
for (x=1; x<=max; x++) {
    temp=x;
    prods=prods*(double)temp/((double)temp+s);
    c=-s/2.0;
    if (c>=0.0)
        temp1=pow((double)(x+1),c);
    else {
        temp1=pow((double)(x+1),-c);
        temp1=1.0/temp1;
        }
```

```
    temps=temp1*(cos(t*log(x+1)));
    tempt=temp1*(sin(t*log(x+1)));
    a=prods*temps-tempt;
    b=prods*tempt+temps;
    if (s>=0.0)
        temp1=pow((double)x,s);
    else {
        temp1=pow((double)x,-s);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(x)));
    tempt=temp1*(sin(t*log(x)));
    temp1=temps*temps+tempt*tempt;
    c=temps/temp1;
    d=tempt/temp1;
    sums=sums+c;
    sumt=sumt-d;
    temps=a*sums-b*sumt;
    tempt=a*sumt+b*sums;
    c=temps*e-tempt*f;
    d=temps*f+tempt*e;
    temps=c;
    tempt=d;
    if (x>xmin)
        fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
    }
fclose(Outfp);
return;
}

#include <math.h>
#include <stdio.h>
//
// zeta function (when noinc=0)
// X(z,n) (when noinc=1)
//
unsigned int max=1000;
double s=0.50;
//double t=14.13472514173470;
```

```
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
unsigned int xmin=0; // usually set to 0
unsigned int noinc=0; // if set, don't add additional term
unsigned int out=1; // set to 1 when not finding inflection points
void main() {
unsigned int x;
double temp1,temps,tempt,sumr,sumi,a,b,olds,oldt;
FILE *Outfp;
Outfp = fopen("spiral1.dat","w");
sumr=0.0;
sumi=0.0;
olds=0.0;
oldt=0.0;
for (x=1; x<=max; x++) {
    if (s>=0.0)
        temp1=pow((double)x,s);
    else {
        temp1=pow((double)x,-s);
        temp1=1.0/temp1;
        }
    a=temp1*(cos(t*log(x)));
    b=temp1*(sin(t*log(x)));
    temp1=a*a+b*b;
    sumr=sumr+a/temp1;
    sumi=sumi-b/temp1;
    if (s>=0.0)
        temp1=pow((double)max,s);
    else {
        temp1=pow((double)max,-s);
        temp1=1.0/temp1;
        }
```

```
    temps=temp1*(cos(t*log(max)));
    tempt=temp1*(sin(t*log(max)));
    if (noinc==0) {
        temps=sumr+temps/2.0;
        tempt=sumi+temps/2.0;
        }
    else {
        temps=sumr;
        tempt=sumi;
        }
    if (x>xmin) {
        if (out==1)
            fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
        if ((out==2)&&((olds>0.0)&&(temps<0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==3)&&((oldt>0.0)&&(tempt<0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==4)&&((olds<0.0)&&(temps>0.0)&&(x>1)))
            fprintf(Outfp,"                                    if
((out==5)&&((oldt<0.0)&&(tempt>0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        olds=temps;
        oldt=tempt;
        }
    }
printf(" %.10llf %.10llf \n",temps,tempt);
fclose(Outfp);
return;
}
```

```
#include <math.h>
#include <stdio.h>
//
// Y(z,n)
//
unsigned int maxn=1000;
unsigned int out=1;
double s=0.50;
//double t=14.13472514173470;
```

```
//double t=21.02203963877156;
//double t=25.01085758014569;
double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
void main() {
double temp1,temps,tempt,x,olds,oldt;
unsigned int n,max;
FILE *Outfp;
Outfp = fopen(”spiral2.dat”,”w”);
olds=0.0;
oldt=0.0;
for (n=1; n<=maxn; n++) {
    max=n;
    x=1.0-s;
    if (x>=0.0)
        temp1=pow((double)max,x);
    else {
        temp1=pow((double)max,-x);
        temp1=1.0/temp1;
        }
    temps=temp1*(x*cos(t*log(max))+t*sin(t*log(max)));
    temps=temps/(x*x+t*t);
    tempt=temp1*(t*cos(t*log(max))-x*sin(t*log(max)));
    tempt=tempt/(x*x+t*t);
    if (out==1)
        fprintf(Outfp,” %.10llf %.10llf \n”,temps,tempt);
    if ((out==2)&&((olds>0.0)&&(temps<0.0)&&(n>1)))
        fprintf(Outfp,” %d %.10llf %.10llf \n”,n,temps,tempt);
    if ((out==3)&&((oldt>0.0)&&(tempt<0.0)&&(n>1)))
        fprintf(Outfp,” %d %.10llf %.10llf \n”,n,temps,tempt);
    if ((out==4)&&((olds<0.0)&&(temps>0.0)&&(n>1)))
        fprintf(Outfp,” %d %.10llf %.10llf \n”,n,temps,tempt);
    if ((out==5)&&((oldt<0.0)&&(tempt>0.0)&&(n>1)))
        fprintf(Outfp,” %d %.10llf %.10llf \n”,n,temps,tempt);
```

```
    olds=temps;
    oldt=tempt;
    }
printf(" %.10llf %.10llf \n",temps,tempt);
fclose(Outfp);
return;
}

#include <math.h>
#include <stdio.h>
//
// C-transformation
//
unsigned int max=1000;
double s=0.50;
double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
unsigned int out=1; // use out=3 otherwise
void main() {
unsigned int x,z;
double temp1,temps,tempt,sumr,sumi,a,b,olds,oldt,y;
FILE *Outfp;
Outfp = fopen("ctrans.dat","w");
sumr=0.0;
sumi=0.0;
olds=0.0;
oldt=0.0;
for (x=1; x<=max; x++) {
    z=max;
    if (s>=0.0)
        temp1=pow((double)z,s);
```

```
    else {
        temp1=pow((double)z,-s);
        temp1=1.0/temp1;
        }
    a=temp1*(cos(t*log(z)));
    b=temp1*(sin(t*log(z)));
    temp1=a*a+b*b;
    sumr=sumr+a/temp1;
    sumi=sumi-b/temp1;
    y=1-s;
    if (y>=0.0)
        temp1=pow((double)max,y);
    else {
        temp1=pow((double)max,-y);
        temp1=1.0/temp1;
        }
    temps=temp1*(y*cos(t*log(max))+t*sin(t*log(max)));
    tempt=temp1*(t*cos(t*log(max))-y*sin(t*log(max)));
    temps=temps/(y*y+t*t);
    tempt=tempt/(y*y+t*t);
    temps=sumr+temps;
    tempt=tempt-sumi;
    if (out==1)
        fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
    if ((out==2)&&((olds>0.0)&&(temps<0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==3)&&((oldt>0.0)&&(tempt<0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==4)&&((olds<0.0)&&(temps>0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==5)&&((oldt<0.0)&&(tempt>0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    olds=temps;
    oldt=tempt;
    }
printf(" %.10llf %.10llf \n",temps,tempt);
fclose(Outfp);
return; }
```

REFERENCES

[1] Caceres, Pedro."$\zeta(z) = X(z) - Y(z)$ A decomposition of the Riemann Zeta Function for $Re(z) > 0$, $z \neq 1$", 2020. VIXRA:https://vixra.org/2003.0189, accessed March 2020.

[2] Caceres, Pedro, Proof of the Riemann Hypothesis using the decomposition Zeta(z)=X(z)-Y(z), RG/339841648, 2020