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The C-Transformation of the Riemann Zeta Function

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Abstract

Caceres derived the C-transformation of the Riemann zeta function and defined the X(z,n) and Y(z,n) functions. Variants of these functions are introduced. This is relevant to his work on proving the Riemann hypothesis.
Keywords: Riemann zeta function, Riemann hypothesis

1. INTRODUCTION

Caceres [1] defined the C-transformation as

$$C_n\{f\} = \sum_{k=1}^{n} f(k) - \int f(n)dn \tag{1}$$

and derived the following function

$$X(z,n) = (\sum_{k=1}^{n} k^{-\alpha}(\cos(\beta \cdot \ln(k)) + \frac{1}{2}n^{-\alpha}\cos(\beta\ln(n)) + \tag{2}$$

$$i(\sum_{k=1}^{n} k^{-\alpha}(\sin(\beta \cdot \ln(k)) + \frac{1}{2}n^{-\alpha}\sin(\beta\ln(n))). \tag{3}$$

The associated function is

$$Y(z,n) = n^{(1-\alpha)}\frac{1}{[(1-\alpha)^2+\beta^2]}[((1-\alpha)\cdot\cos(\beta\ln(n)) + \beta\cdot\sin(\beta\ln))) + \tag{4}$$

$$i(\beta\cdot\cos(\beta\ln(n)) - (1-\alpha)\cdot\sin(\beta\ln(n)))]. \tag{5}$$

$X(z) = \lim_{n\to\infty} X(z,n)$ and $Y(z) = \lim_{n\to\infty} Y(z,n)$. Caceres proved the following

Theorem 1. $\zeta(z) = X(z) - Y(z)$.

A plot of $X(z,n)$ and $\sum_{k=1}^{n} \frac{1}{n^z}$ (the Riemann zeta function) for $z = (0.5, 14.13472514173470)$ (the first non-trivial zeta function zero) and $n \leq 1000$ and

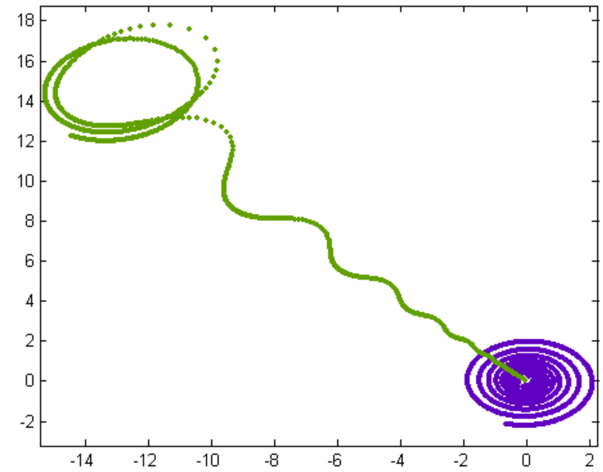

Figure 1

For zeta function zeros, the logarithmic spiral is centered on (0, 0). The offset calculated between the ends of the curves is (-14.4574737406, 12.2614653997).

Let $E(z,n)$ denote the end points of the logarithmic spirals generated by the zeta function. $E(z,n)$ is a logarithmic spiral in itself. A plot of $E(z,n)$ and $Y(z,n)$ for the tenth zeta function zero (0.5, 49.77383247767230) is

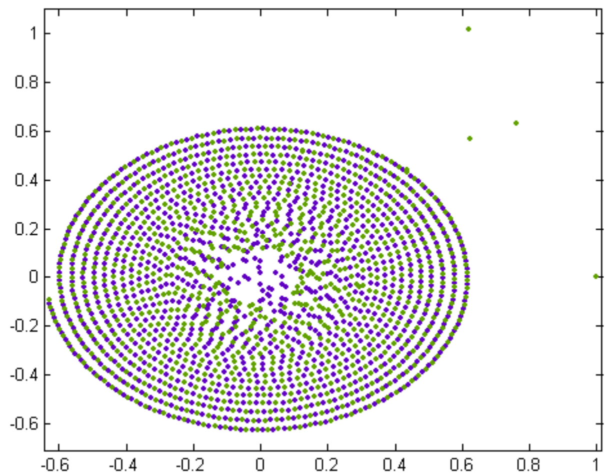

Figure 2

The curves mainly overlap. For z values not equal to zeta function zeros, the curves don't overlap.

Let $\zeta_n'(z)$ denote $\sum_{k=1}^{n}\frac{1}{(n-k+1)^z}$. A plot of $E(z,n)$ and $\zeta_n'(z)$ for the first zeta function zero is

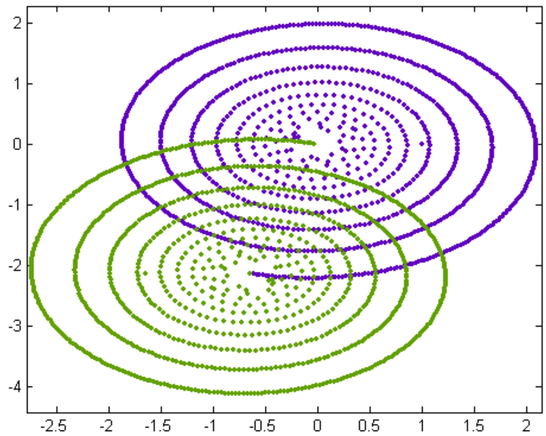

Figure 3

Let $X'(z,n)$ denote

$$(\sum_{k=1}^{n}(n-k+1)^{-\alpha}(\cos(\beta\cdot\ln(k))+\frac{1}{2}n^{-\alpha}\cos(\beta\ln(n))+ \tag{6}$$

$$i(\sum_{k=1}^{n}(n-k+1)^{-\alpha}(\sin(\beta\cdot\ln(k))+\frac{1}{2}n^{-\alpha}\sin(\beta\ln(n))). \tag{7}$$

A plot of $X(z,n)$ and $X'(z,n)$ for the first zeta function zero is

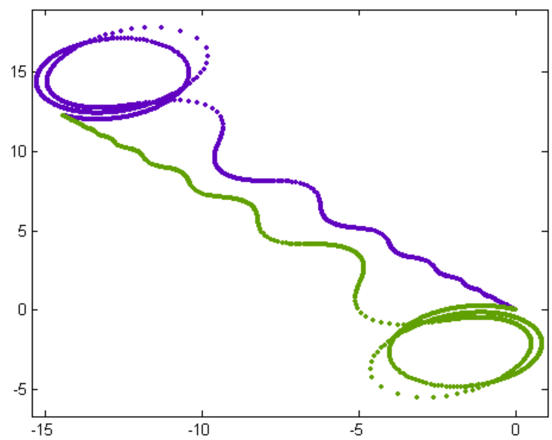

Figure 4

Let $Y'(z,n)$ denote $C_n\{\frac{1}{(n-k+1)^z}\}$ where $k = 1$ to n (see [24] and [25] in Caceres' article for $C_n\{\frac{1}{x^z}\}$). A plot of $Y(z,n)$ and $Y'(z,n)$ for the first zeta function zero is

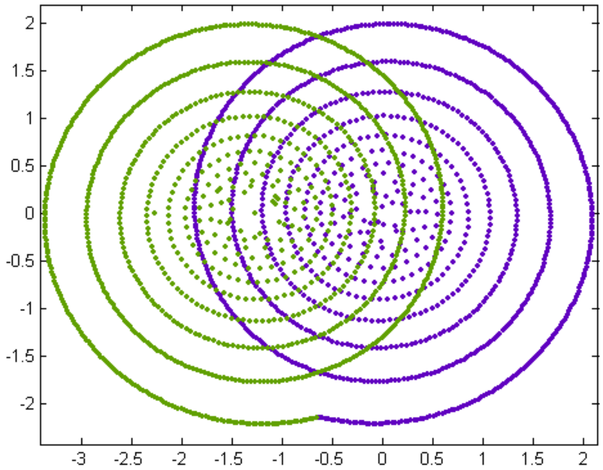

Figure 5

2. ABSOLUTE SQUARES

A "polynomial" form of $Y(z,n)$ (derived by Caceres) is

$$|Y(z,n)|^2 \geq n^{2(1-a)}/[(1-\alpha)^2 + \beta^2] \tag{8}$$

A plot $|Y(z,n)|^2$ and the real and imaginary components of the right-hand side of the inequality for the first zeta function zero and $n \leq 1000$ is

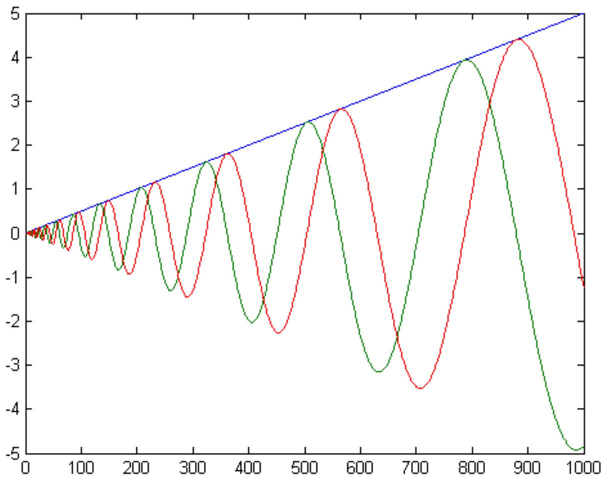

Figure 6

The slope of $|Y(z,n)|^2$ is

$$d(|Y(z,n)|^2)/dn = 2(1-\alpha)n^{1-2\alpha}/[(1-\alpha)^2+\beta^2] \tag{9}$$

A plot $|Y(z,n)|^2$ and the real and imaginary components of the right-hand side of the inequality for $z=(0.5,34.0)$ is

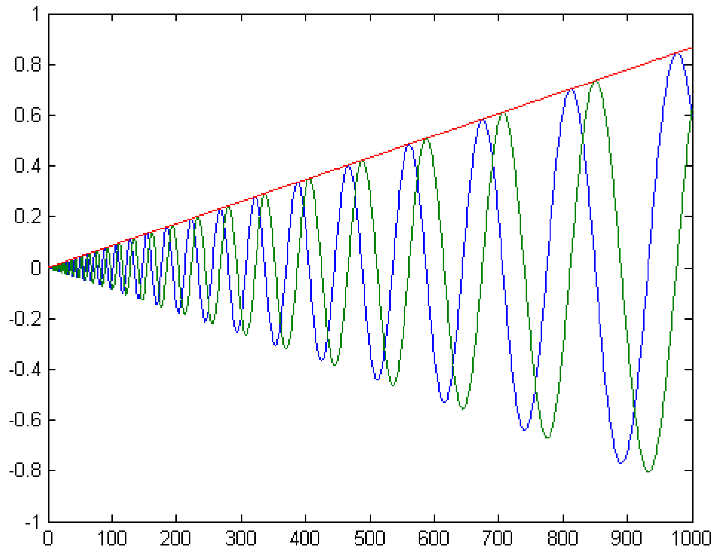

Figure 7

The slopes of the two above lines are 0.004999 and 0.0008649 (relative to n). The ratio of these two slopes is about 5.78. The ratio of $(1-0.5)^2+34.0^2$ and $(1-0.5)^2+14.134725^2$ is about 5.78.

Caceres [2] proved the following

Theorem 2. *$|Y(z,n)|^2$ is a straight line if and only if $\alpha=1/2$.*

A "polynomial" form of $Y'(z,n)$ is

$$|Y'(z,n)|^2 > (n-k+1)^{2(1-a)}/[(1-\alpha)^2+\beta^2] \tag{10}$$

A plot $|Y'(z,n)|^2$ and the real and imaginary components of the right-hand side of the inequality for the first zeta function zero and $n \le 10000$ is

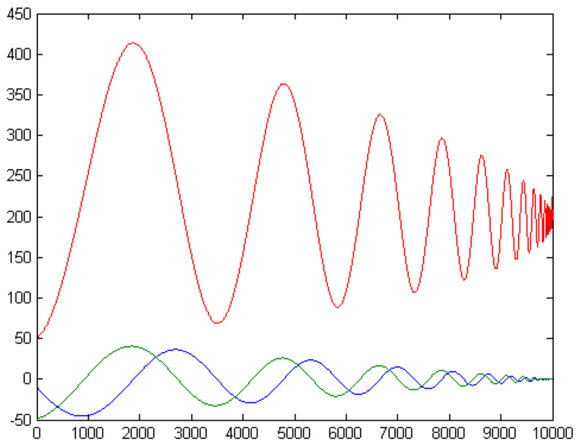

Figure 8

The peaks and valleys of $|Y'(z,n)|^2$ align with the peaks and valleys of the imaginary components of the right-hand side of the inequality. The two curves have slopes of zero at the same places.

A plot $|Y'(z,n)|^2$ and the real and imaginary components of the right-hand side of the inequality for $z = (0.5, 34.0)$ and $n \leq 10000$ is

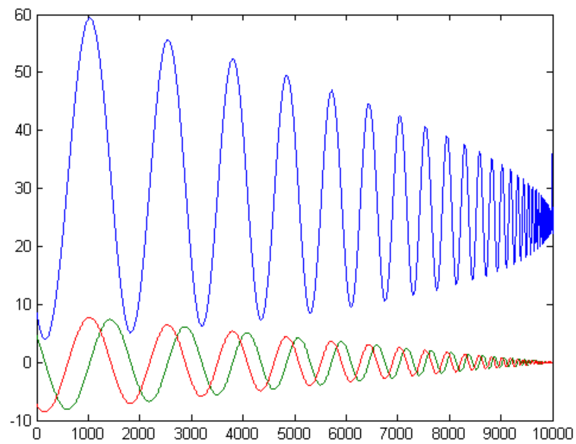

Figure 9

The relationship between the peaks and valleys of the two curves appears to hold for z values with a real part of 1/2.

A plot of the real and imaginary components of the right-hand side of the inequality for $z = (0.4, 14.13472514)$ is

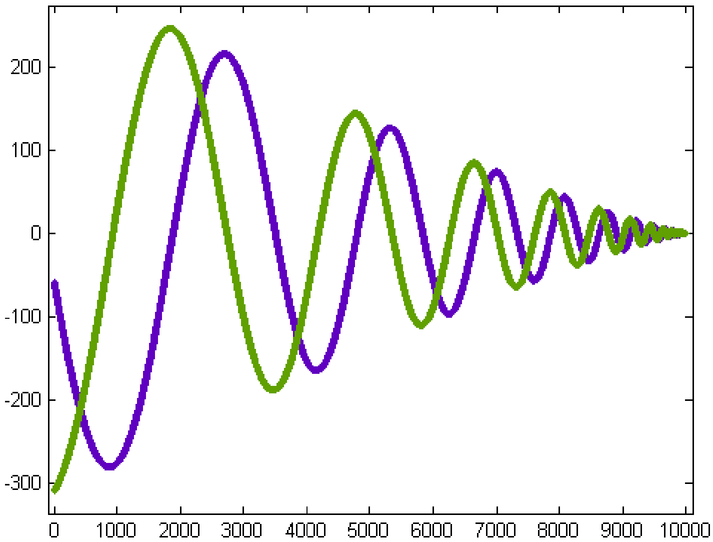

Figure 10

Unlike a real part of 1/2, the maxima and minima of the curves are not on a straight line.

A plot of $|Y'(z,n)|^2$ and $|Y(z,n)|^2$ for the third zeta function zero and $n \leq 2000$ is

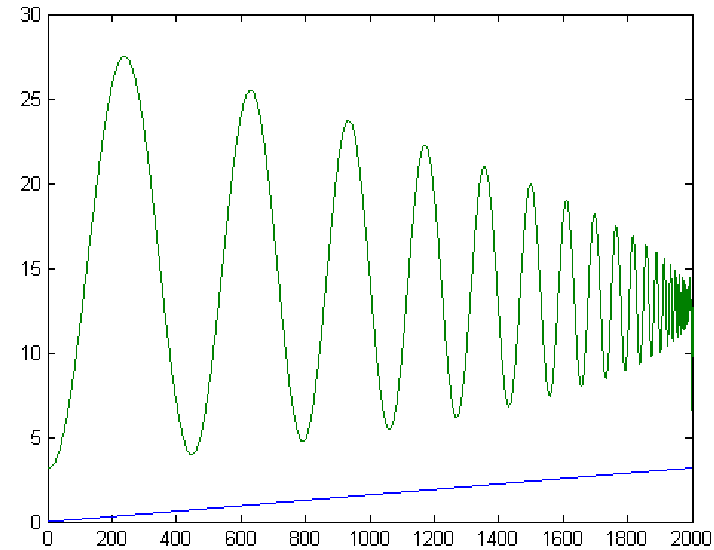

Figure 11

A plot of the element-by-element ratio of these values is

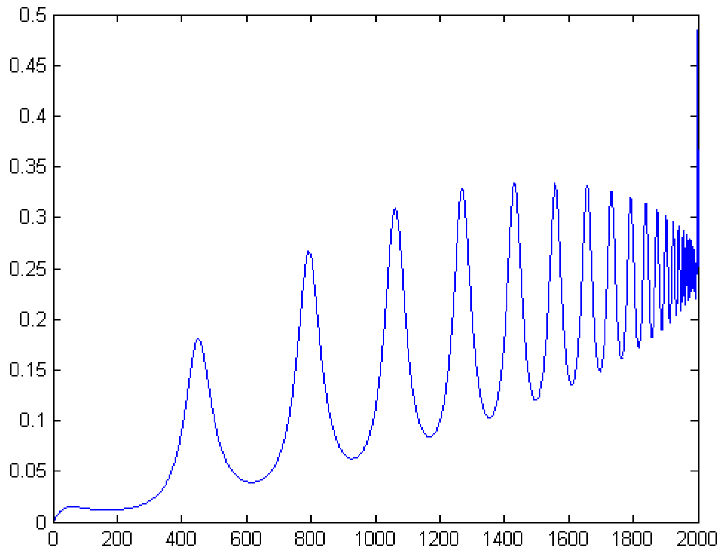

Figure 12

The values appears to converge to 1/4. Caceres proved the following

Theorem 3. *When* $\alpha = 1/2$, $|Y(z, n)|^2 = \frac{n}{]\beta^2+\frac{1}{4}]}$

A plot of $n^{2(1-a)}/[(1-\alpha)^2 + \beta^2]$ for the tenth zeta function zero is

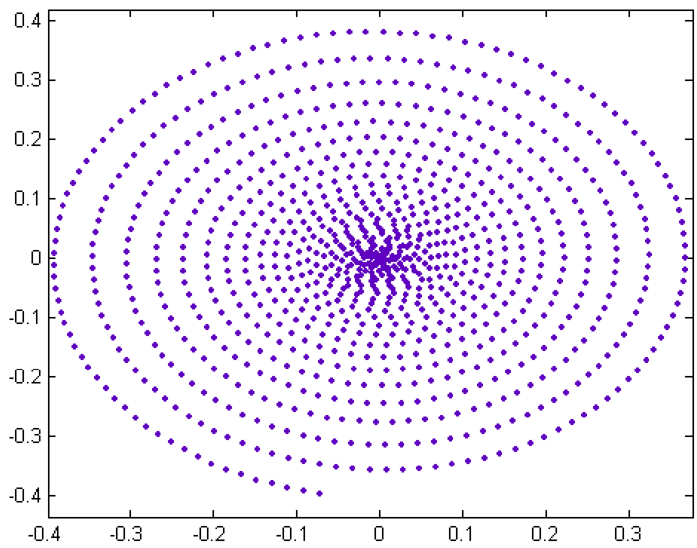

Figure 13

This is not a logarithmic spiral. A plot of the inflection points is

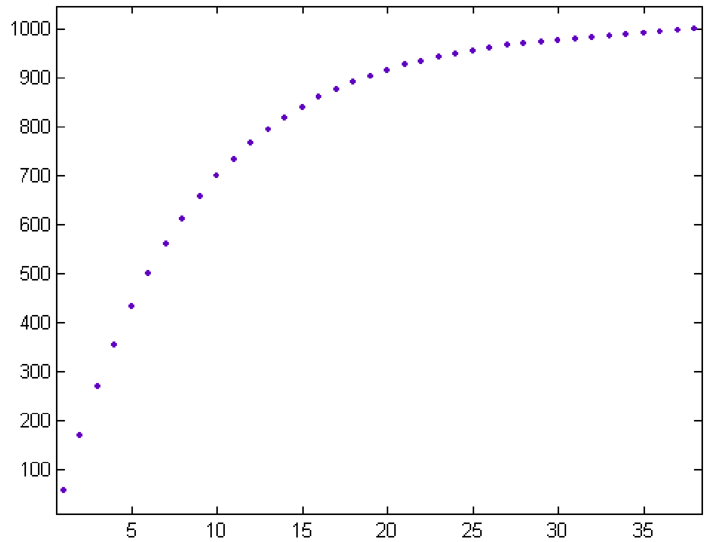

Figure 14

A plot of the inflection points versus the logarithms of the values along the x-axis is

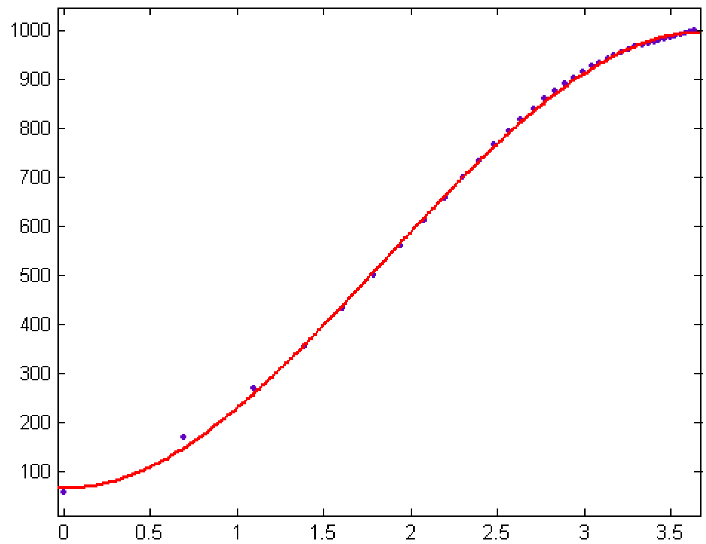

Figure 15

This is a cubic curve.

A plot of $|X(z,n)|^2$ and $|X'(z,n)|^2$ for the tenth zeta function zero and $n \leq 2000$ is

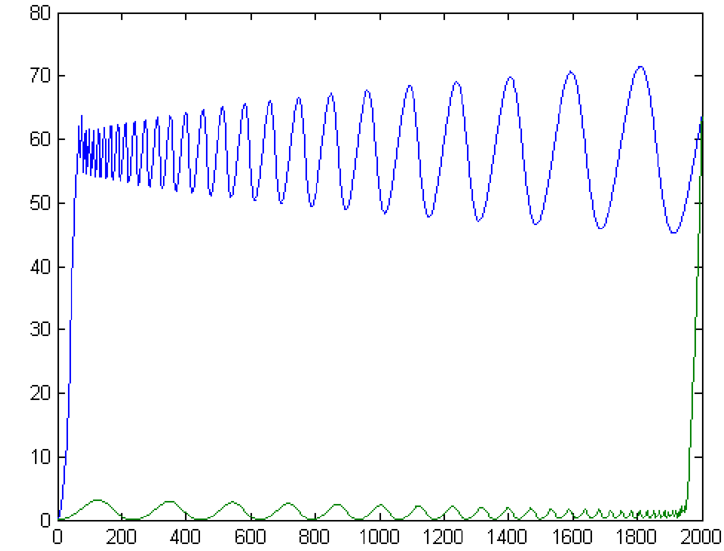

Figure 16

A plot of the element-by-element product of these values is

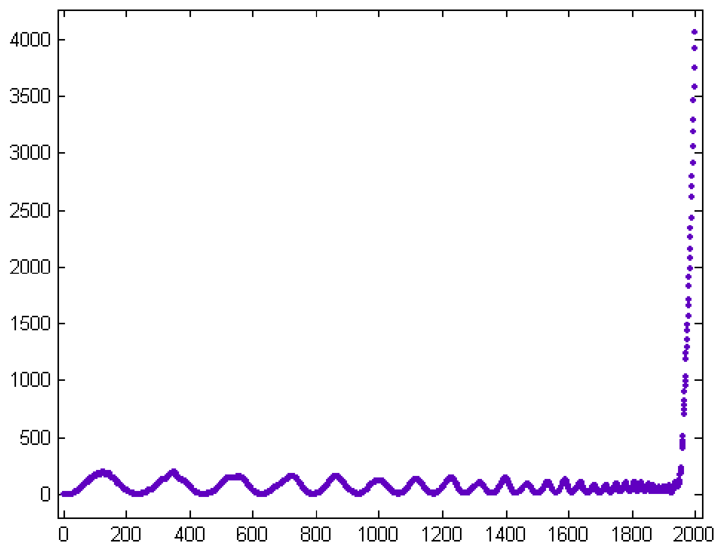

Figure 17

For $n \leq 10000$, the values of $|\zeta_n(z)|^2$ for the first ten zeta function zeros are 49.9924, 22.6166, 15.9805, 10.8006, 9.2173, 7.0776, 5.9719, 5.3265, 4.3391, and 4.0362. The values of $|\zeta_n'(z)|^2$ are the same. A plot of these values versus the logarithms of the zeta function zeros is

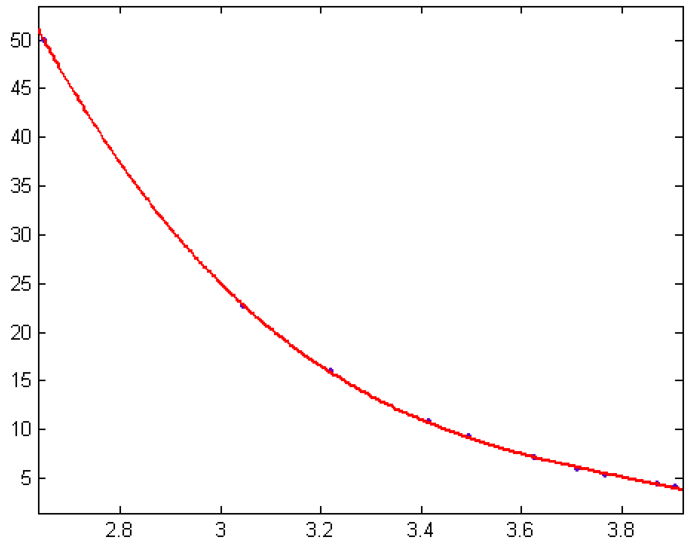

Figure 18

This is a cubic curve.

3. METHODS

The C code for computing X'(z,n), and Y'(z,n) is as follow.

```
#include <math.h>
#include <stdio.h>
//
// zeta function variant (when noinc=1)
// X'(z,n) (when noinc=0)
//
unsigned int max=10000;
double s=0.50;
//double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
```

```
double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
unsigned int xmin=0; // usually set to 0
unsigned int noinc=0; // if set, don't add additional term
unsigned int out=1; // set to 1 when not finding inflection points
// set to 3 when noinc=0
// set to 3 or 4 when noinc=1
void main() {
unsigned int x,y;
double temp1,temps,tempt,sumr,sumi,a,b,olds,oldt;
FILE *Outfp;
Outfp = fopen("spirala.dat","w");
sumr=0.0;
sumi=0.0;
olds=0.0;
oldt=0.0;
for (x=1; x<=max; x++) {
    y=max-x+1;
    if (s>=0.0)
        temp1=pow((double)y,s);
    else {
        temp1=pow((double)y,-s);
        temp1=1.0/temp1;
        }
    a=temp1*(cos(t*log(y)));
    b=temp1*(sin(t*log(y)));
    if (s>=0.0)
        temp1=pow((double)max,s);
    else {
        temp1=pow((double)max,-s);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(max)));
    tempt=temp1*(sin(t*log(max)));
    if (noinc==0) {
```

```
        a=a+temps/2.0;
        b=b+temps/2.0;
        }
    temp1=a*a+b*b;
    sumr=sumr+a/temp1;
    sumi=sumi-b/temp1;
    temps=sumr;
    tempt=sumi;
    if (x>xmin) {
        if (out==1)
            fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
        if ((out==2)&&((olds>0.0)&&(temps<0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==3)&&((oldt>0.0)&&(tempt<0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==4)&&((olds<0.0)&&(temps>0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==5)&&((oldt<0.0)&&(tempt>0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        olds=temps;
        oldt=tempt;
        }
    }
printf(" %.10llf %.10llf \n",temps,tempt);
fclose(Outfp);
return;
}

#include <math.h>
#include <stdio.h>
//
// Y'(z,n)
//
unsigned int max=2000;
double s=0.50;
//double t=14.13472514173470;
//double t=21.02203963877156;
double t=25.01085758014569;
//double t=30.42487612585951;
```

```
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
unsigned int out=1; // set to 3 for inflection points
void main() {
unsigned int x,z;
double temp1,temps,tempt,sumr,sumi,a,b,olds,oldt,y;
FILE *Outfp;
Outfp = fopen("spiralc.dat","w");
sumr=0.0;
sumi=0.0;
olds=0.0;
oldt=0.0;
for (x=1; x<=max; x++) {
    z=max-x+1;
    if (s>=0.0)
        temp1=pow((double)z,s);
    else {
        temp1=pow((double)z,-s);
        temp1=1.0/temp1;
        }
    a=temp1*(cos(t*log(z)));
    b=temp1*(sin(t*log(z)));
    temp1=a*a+b*b;
    sumr=sumr+a/temp1;
    sumi=sumi-b/temp1;
    y=1-s;
    if (y>=0.0)
        temp1=pow((double)max,y);
    else {
        temp1=pow((double)max,-y);
        temp1=1.0/temp1;
        }
    temps=temp1*(y*cos(t*log(max))+t*sin(t*log(max)));
    tempt=temp1*(t*cos(t*log(max))-y*sin(t*log(max)));
    temps=temps/(y*y+t*t);
```

```
    tempt=tempt/(y*y+t*t);
    temps=sumr+temps;
    tempt=tempt-sumi;
    if (out==1)
        fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
    if ((out==2)&&((olds>0.0)&&(temps<0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==3)&&((oldt>0.0)&&(tempt<0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==4)&&((olds<0.0)&&(temps>0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==5)&&((oldt<0.0)&&(tempt>0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    olds=temps;
    oldt=tempt;
    }
printf(" %.10llf %.10llf \n",temps,tempt);
fclose(Outfp);
return;
}
```

REFERENCES

[1] Caceres, Pedro."$\zeta(z) = X(z) - Y(z)$ A decomposition of the Riemann Zeta Function for $Re(z) > 0$, $z \neq 1$", 2020. VIXRA:https://vixra.org/2003.0189, accessed March 2020.

[2] Caceres, Pedro, Proof of the Riemann Hypothesis using the decomposition Zeta(z)=X(z)-Y(z), RG/339841648, 2020